Functions

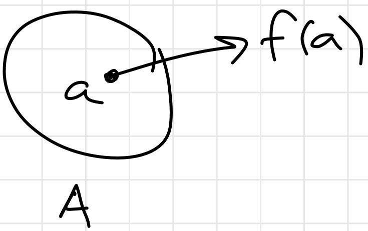
Def Let A, B be sets.

$f: A \to B$ is a function if f assigns to each $a \in A$ a single value $b \in B$, denoted $f(a)$.

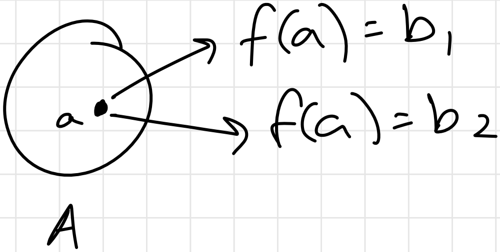
"f from A to B"

Equivalently, f has 3 properties:

1) for each $a \in A$, $f(a)$ is defined.

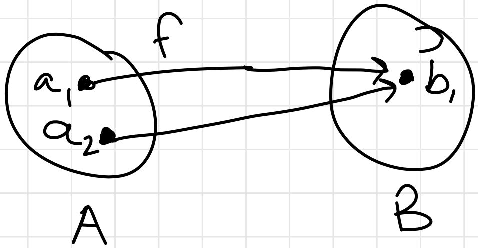


2) For each $a \in A$, $f(a)$ does not produce 2 different outputs.



only if $b_1 = b_2$

3) for each $a \in A$, $f(a) \in B$.



$f: A \to B$

A is the <u>domain</u> of f

B is the <u>codomain</u> of f

The <u>range</u> of f is $\{f(a) : a \in A\}$

range $\subseteq$ codomain

let $A = \{1, 2, 3\}$
let $B = \{x, y\}$

$a \in A$	$b \in B$
1	$x = f(1)$
2	$y = f(2)$
3	$x = f(3)$

Props:

(1) $\forall a \in A$, $f(a)$ is defined ✓

(2) $\forall a \in A$, $f(a)$ does not produce 2 diff. outputs ✓

(3) $\forall a \in A$, $f(a) \in B$ ✓

- exactly 1 row for every element of A
- some elements of B can have zero rows, or elements of B can have multiple rows

ex $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

domain: $\mathbb{R}$
codomain: $\mathbb{R}$
range: $\mathbb{R}^{\geq 0}$ (reals greater than or equal to 0)

Intuitive "proof" of 3 properties:

(1) $\forall x \in \mathbb{R}$, $f(x) = x^2$ ✓

(2) $\forall x \in \mathbb{R}$, $f(x) = x^2$, a single value

(3) $\forall x \in \mathbb{R}$, $f(x) \in \mathbb{R}$, because $x^2 \in \mathbb{R}$

ex $f: \mathbb{R} \to \mathbb{R}^{<0}$, $f(x) = x^2$

f is not a function.

Violates (3). Consider $2 \in \mathbb{R}$. $f(2) = 4 \notin \mathbb{R}^{<0}$

ex $s: \mathbb{Z} \to \mathbb{Z}$ defined by $s(x) = x+1$

"successor function"

domain, codomain: $\mathbb{Z}$

range: $\mathbb{Z}$

<u>claim</u> $s: \mathbb{Z} \to \mathbb{Z}$ is a function.

<u>Proof</u> We prove all 3 properties.

1) $\forall x \in \mathbb{Z}$, $s(x)$ is defined as $x+1$.

2) To show $\forall x \in \mathbb{Z}$, $s(x)$ does not produce 2 diff. outputs, we show that if $s(x) = a$ and $s(x) = b$, then $a = b$.

Assume $s(x) = a$ and $s(x) = b$.

$a = x+1$, $b = x+1$ def. of s

$a = b$ substitution

3) WTS (want to show) $\forall x \in \mathbb{Z}$, $s(x) \in \mathbb{Z}$.

$s(x) = x+1$, which is an integer because int + int = int.

Examples from last time:

1. $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(a) = 5$

Properties:

1) Defined $\forall x \in \mathbb{Z}$: yes. $g(x) = 5$.
2) $\forall x \in \mathbb{Z}$, $g(x)$ maps to only one output.

Proof:

let $x \in \mathbb{Z}$ and $g(x) = a$ and $g(x) = b$.

$a = 5$, $b = 5$ def. of $g(x)$

$a = b$ substitution. $\square$

3) $\forall x \in \mathbb{Z}$, $g(x) \in \mathbb{Z}$. Yes - $g(x) = 5$ $\forall x \in \mathbb{Z}$.

Domain: $\mathbb{Z}$
Codomain: $\mathbb{Z}$
Range: $\{5\}$

2. $E : \mathbb{Z} \to \{T, F\}$ defined by $E(x) = \begin{cases} T & x \text{ is even} \\ F & x \text{ is odd} \end{cases}$

Properties:

1) Defined $\forall x \in \mathbb{Z}$: yes.
2) $\forall x \in \mathbb{Z}$, $E(x)$ maps to only one output.

Proof:

Let $x \in \mathbb{Z}$. WTS that if $E(x) = a$ and $E(x) = b$, then $a = b$.

We prove using cases.

Case 1: x is even.

Let $E(x) = a$ and $E(x) = b$. Since x is even, $a = T$ and $b = T$, so $a = b$.

Case 2: x is odd.

Let $E(x) = a$ and $E(x) = b$. Since x is odd, $a = F$ and $b = F$, so $a = b$.

Since the claim is true in all cases, the claim is true.

3) $\forall x \in \mathbb{Z}$, $E(x) \in \{T, F\}$. Yes.

Domain: $\mathbb{Z}$
Codomain: $\{T, F\}$
Range: $\{T, F\}$

3. $p : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ defined by $p(x) = x - 1$

Not a function! Fails property 3.

For $x = 0 \in \mathbb{Z}^{\geq 0}$, $p(x) = x - 1 = 0 - 1 = -1 \notin \mathbb{Z}^{\geq 0}$.

4. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x)$ the number whose absolute value is x.

Not a function! Fails property 2.

For $x = 5 \in \mathbb{Z}$, the number whose abs. val. is 5 is both -5 and 5.

Violates property 1.

Consider $x = -5$. It's undefined which number has absolute value -5.

Another example:

$f : \mathbb{R} \to \mathbb{R}$ $f(x) = \frac{1}{x}$

1) $\forall x \in \mathbb{R}$, $f(x)$ is defined.

Consider $x = 0$. $f(x) = \frac{1}{x}$ is not defined.

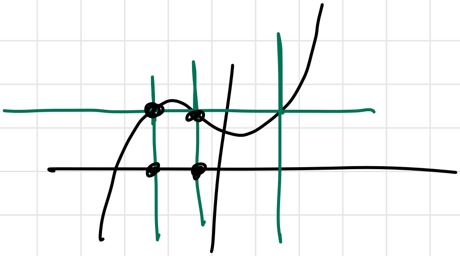
Def A function $f: A \to B$ is

1. onto (surjective) if
$\forall b \in B \ \exists a \in A : f(a) = b$
$\equiv \forall b \in B$, something in A maps to it
$\equiv \forall b \in B$, b shows up in at least 1 row of the table
$\equiv$ codomain = range

ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x$

2. one-to-one (injective) if
1:1

$\forall a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
$\equiv \forall b \in B$, at most 1 thing in A maps to it
$\equiv \forall b \in B$, b shows up in at most 1 row of the table!

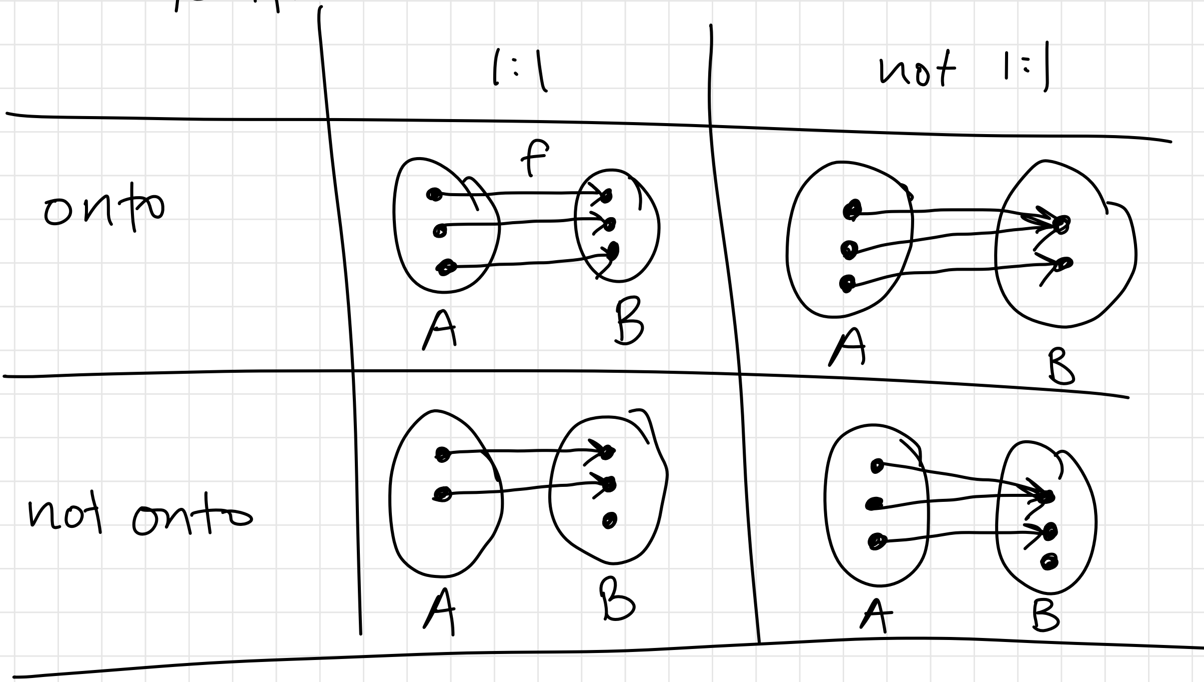


ex $f: \mathbb{R} \to \mathbb{R} : f(x) = x^2$

$f(-2) = 4$
$f(2) = 4$

3. a bijection if onto and 1:1

$\equiv \forall b \in B$, exactly 1 elt. of A maps to it.



How do we prove that f is onto or 1:1 ?

Onto

WTS $\forall b \in B \; \exists a \in A : f(a) = b$.

$\equiv$ If $b \in B$, then $\exists a \in A : f(a) = b$.

Step 1: Assume $b \in B$.

Step 2: Construct a s.t. $f(a) = b$.

ex $s: \mathbb{Z} \to \mathbb{Z}$ defined by $s(x) = x+1$.

Claim: s is onto.

example: $b = 7$. What is $a \in \mathbb{Z}$ s.t. $f(a) = b = 7$?

$a = 6$, $f(a) = 6+1 = 7$.

Proof: Assume $b \in \mathbb{Z}$. Want to construct $a \in \mathbb{Z}$ s.t. $f(a) = b$.
Consider $a = b-1$. $a \in \mathbb{Z}$ since int − int = int, and $s(a) = (b-1)+1$ by def. of s, so $s(a) = b$, as needed.

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A, B sets

Review definitions: $f: A \to B$

$\forall, \exists$

onto: everything in codomain is mapped to

$$\forall b \in B : \exists a \in A : f(a) = b$$

1:1

each domain value has a unique codomain value

$$\forall a_1, a_2 \in A : (a_1 \neq a_2) \Rightarrow (f(a_1) \neq f(a_2))$$

warmup: $A = \{0, 1, 2\}$ $B = \{3, 4\}$

Example 2.57: Sample onto/non-onto functions.

Let $A = \{0, 1, 2\}$ and $B = \{3, 4\}$. Give an example of a function that satisfies the following descriptions; if there's no such function, explain why it's impossible.

$f : A \to B$

$f(0) = 3$

$f(1) = 3$

$f(2) = 4$

1 an onto function $f : A \to B$.

2 a function $g : A \to B$ that is *not* onto.

3 an onto function $h : B \to A$.

(1) give an onto function ✓

$f : A \to B$

(2) give ~~an~~ a not onto function

$g : A \to B$ $f(a) = 3$ $\forall a \in A$

(3) give an onto function

$h : B \to A$

$f(3) = 0$

$f(4) = 1$

(or say why can't)

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not onto: WTS $\neg(\forall b \in B : \exists a \in A : f(a) = b)$

$\equiv \exists b \in B : \forall a \in A : f(a) \neq b$

reals $\qquad \mathbb{Q}$ = rations

Claim:

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is onto. $\qquad$ T/F ?

Proof:

let $b \in \mathbb{R}$. WTS $\exists a \in \mathbb{R} : f(a) = b$. let $a = \sqrt{b}$.

$\to$ $\boxed{\sqrt{b} \in \mathbb{R}}$ false $\qquad b \in \mathbb{R}$

$f(a) = (\sqrt{b})^2$ $\qquad$ def. of f

$f(a) = b$ $\qquad$ def of $(\sqrt{})^2$

Is this proof valid? No!

Claim: f is not onto.

codomain $\qquad$ domain,

Proof: Consider $b = -1 \in \mathbb{R}$. WTS $\forall a \in \mathbb{R}$, $f(a) \neq -1$.

$\forall a \in \mathbb{R} : f(a) = a^2$ $\qquad$ def. of f

$\forall a \in \mathbb{R} : f(a) \geq 0$ $\qquad$ property of ^{2}

$\forall a \in \mathbb{R} : f(a) \neq b$ $\qquad$ $b < 0$

Note: $f(-1) = 1$, so $f(-1) = -1$

Proving 1:1

WTS $\underbrace{\forall a_1, a_2 \in A : \overbrace{a_1 \neq a_2}^{p} \Rightarrow \overbrace{f(a_1) \neq f(a_2)}^{q}}$

Direct proof:

1. Assume $a_1, a_2 \in A$, $a_1 \neq a_2$
2. Show $f(a_1) \neq f(a_2)$

Contrapositive $\neg q \Rightarrow \neg p$

$\forall a_1, a_2 : f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

1. assume $a_1, a_2 \in A$, $f(a_1) = f(a_2)$
2. Show $a_1 = a_2$

ex $S : \mathbb{Z} \to \mathbb{Z}$ $S(x) = x+1$

Claim: S is 1:1

Proof: we prove the contrapositive. Suppose $a_1, a_2 \in \mathbb{Z}$ and $S(a_1) = S(a_2)$. WTS $a_1 = a_2$

$a_1 + 1 = a_2 + 1$ def. of S

$a_1 = a_2$ algebra

$\square$

Not 1:1

$$\neg(p \Rightarrow q) \equiv \underline{p} \wedge \neg q$$

$$\neg\left(\forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)\right)$$

$$\exists a_1, a_2 \in A : \neg\left[a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)\right]$$

$$\exists a_1, a_2 \in A : \underbrace{a_1 \neq a_2} \wedge \underbrace{f(a_1) = f(a_2)}$$

exists different a_1, a_2 but $f(a_1) = f(a_2)$

<u>ex</u> $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$

Claim: f is not 1:1

proof: by counterexample. Let $a_1 = 2$, $a_2 = -2$.

$f(a_1) = 4$

$f(a_2) = 4$

so $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.