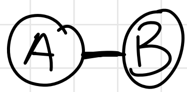
Intro to Graphs

Def An undirected graph $G = (V, E)$ is a non-empty set of vertices (nodes) V and a set $E = \{\{u, v\} : u, v \in V\}$ of edges joining pairs of nodes.

ex

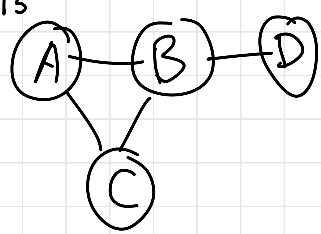
$V = \{A\}$
$E = \emptyset$



$V = \{A, B\}$

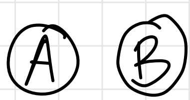
$E = \{\{A, B\}\}$

G_3:



$V = \{A, B, C, D\}$

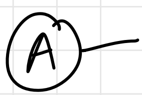
$E = \{\{A, B\}, \{B, D\}, \{B, C\}, \{A, C\}\}$



$V = \{A, B\}$

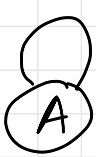
$E = \emptyset$

non-ex



all edges need 2 endpoints

Q



is this a graph? yes.

$V = \{A\}$
$E = \{\{A, A\}\}$
$= \{\{A\}\}$

real-world examples

- Facebook friends

alice — bob
catherine

nodes: people
edge: 2 people are Facebook friends

- blood relationships

Q what property (or properties) would a mathematical relation need to have to be represented as an undirected graph?

ideas: symmetric $a \rightleftarrows b$ $a - b$

reflexive a

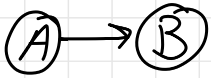
self loops are equivalent when directed or undirected

c c

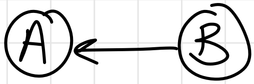
Def A <u>directed graph</u> $G=(V,E)$ has a set of vertices V and edges $E \subseteq V \times V = \{(u,v) : u,v \in V\}$ so that edges are directed from one vertex to another.

Note: relations on a single set and directed graphs are the same!

ex 

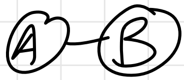
 $V = \{A, B\}$
$E = \{(A, B)\}$

$\neq$

 $V = \{A, B\}$
$E = \{(B, A)\}$

ordered pair
tuple
list
array

undirected:

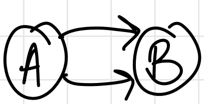
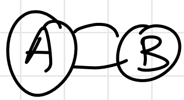
 $E = \{\{A, B\}\}$

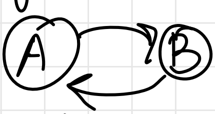
set

real-world example

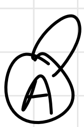
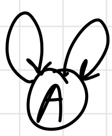
Twitter followers

Def A graph is simple if it contains no parallel edges or self-loops.

parallel edges:  

note that  has no parallel edges

$(A, B) \neq (B, A)$

self-loops:   

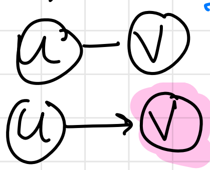
Example 11.3: Self-loops and parallel edges.

Suppose that we construct a graph to model each of the following phenomena. In which settings do self-loops or parallel edges make sense?

1. A social network: nodes correspond to people; (undirected) edges represent friendships.
2. The web: nodes correspond to web pages; (directed) edges represent links.
3. The flight network for a commercial airline: nodes correspond to airports; (directed) edges denote flights scheduled by the airline in the next month.
4. The email network at a college: nodes correspond to students; there is a (directed) edge $\langle u, v \rangle$ if u has sent at least one email to v within the last year.

	self-loops	parallel edges
social network	no	no
the web	yes	yes
flight network	no	yes
email network	yes	no

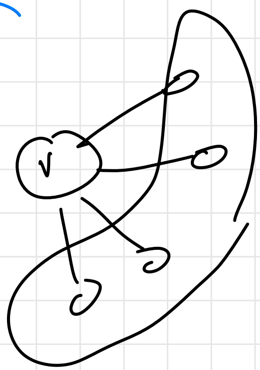
Def Let $e = \{u, v\}$ or (u, v)



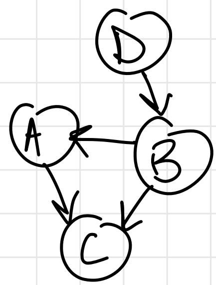
- nodes u, v are <u>adjacent</u> or <u>neighbors</u>
- in a directed graph, v is an <u>out-neighbor</u> of u and u is an <u>in-neighbor</u> of v
- u, v are <u>endpoints</u> of e
- u, v are <u>incident</u> to e

Let v be a node in a simple undirected graph.

$$\text{degree}(v) = \deg(v) = d(v) = \#\text{ of neighbors of } v$$
$$= \left| \{ u \in V : \underbrace{\{v, u\}}_{\text{or } \{u,v\}} \in E \} \right|$$



$\deg(v) = 4$



for directed graphs, $\text{indeg}(v) = \#$ of in-neighbs of v

$\text{outdeg}(v) = \#$ of out-neighbors of v

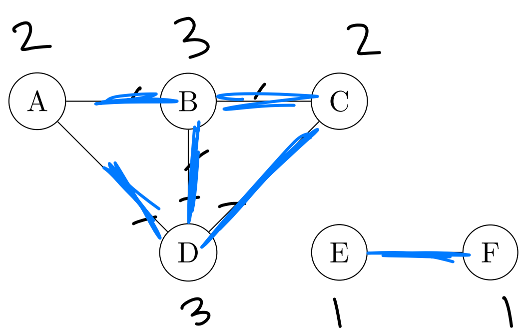
Proofs about graphs

Discrete Structures (CSCI 246)

in-class activity

Names: ____

1. For each of the two graphs, label each node v with $deg(v)$, and give $\sum_{v \in V} deg(v)$, the total degree of the graph, and $|E|$, the number of edges in the graph.

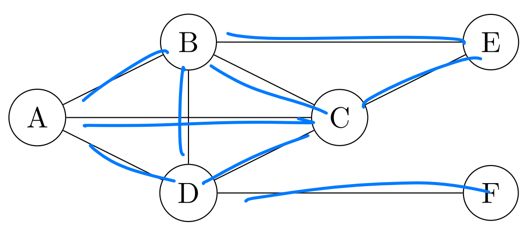


$$\sum_{v \in V} deg(v) = 2+3+2+3+1+1 = 12$$

$$|E| = 6$$

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v)$$

$$2|E| = \sum_{v \in V} deg(v)$$



9 edges

total degree: 18

2. Can you give a conjecture about the relationship between $\sum_{v \in V} deg(v)$ and and $|E|$?

Theorem 11.8 "Handshaking Lemma"

Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof Let $G = (V, E)$ be an undirected graph. Notice that every edge is connected to exactly 2 nodes, meaning that it contributes 1 to the degree of 2 nodes.

So $\sum_{v \in V} \deg(v) = 2|E|$.

Corollary

fact that follows simply from a previous theorem/lemma

Let n_{odd} denote the number of nodes whose degree is odd. Then n_{odd} is even.

Proof Aiming for a contradiction, suppose n_{odd} is odd.

Note that

$$\underbrace{\sum_{v \in V} \deg(v)}_{\text{this is } 2|E|\text{, which is even}} = \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ is odd}}} \deg(v)}_{\text{this must be odd, because sum of odd \# of odds is odd}} + \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ is even}}} \deg(v)}_{\text{this must be even, because sum evens is even}}$$

even = odd + even, ↑

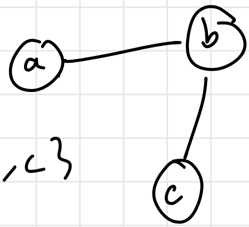
a contradiction!

So n_{odd} must be even. $\square$

"kleek"

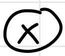
<u>Def</u> A complete graph or clique is an undirected graph $G=(V,E)$ s.t.

$$\forall u,v \in V: \quad u \neq v \implies \{u,v\} \in E$$

is  a clique?

$V=\{a,b,c\}$

No. Consider nodes a, c. $a \neq c$, but $\{a,c\} \in E$

is  a clique?

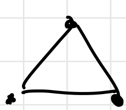
yes.

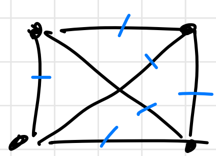
The clique on n nodes is denoted K_n.

examples:

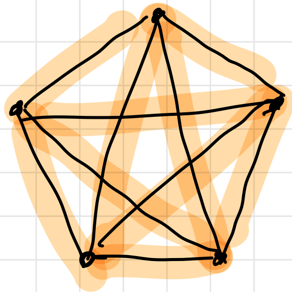
K_1  0

K_2  1

K_3  3

K_4  6

K_5



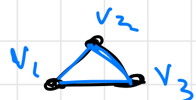
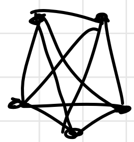
10

Q What is the relationship between $n = |V|$ and $m = |E|$ for K_n?

Conjectures:

$m = (n-1)!$? nope.

$$m = \sum_{i=1}^{n} (i-1) = 0 + 1 + 2 + 3 + \dots + (n-1)$$

n	$\sum_{i=1}^{n}(i-1)$	m	K_n
1	$(1-1) = 0$	0	
2	$(1-1)+(2-1) = 1$	1	
3	$(1-1)+(2-1)+(3-1)$ $= 0+1+2 = 3$	3	
5	$0+1+2+3+4 =$ 10	10	

recall: $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

so $\displaystyle\sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2} = m$, the # edges in K_n.

__claim__ K_n has $\dfrac{n(n-1)}{2}$ edges.

Proof #1 We give a way to count the edges and show that it gives $\frac{n(n-1)}{2}$.

Suppose we have a complete graph K_n. Label its nodes $v_1, v_2, \cdots, v_n$. Starting with v_1, count the uncounted edges adjacent to v_1 and add the count to the total.

v_1 has $n-1$ uncounted edges
v_2 has $n-2$ uncounted edges
$\vdots$
v_{n-1} has 1 uncounted edge
v_n has 0 uncounted edges

$$m = |E| = 0 + 1 + 2 + \cdots + n-1 = \frac{n(n-1)}{2} \qquad \square$$

Proof #2 Let K_n be the complete graph on n nodes.

Note that every node has degree $n-1$.

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (n-1) = n(n-1)$$

But by the handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

$$n(n-1) = 2|E|$$

$$\frac{n(n-1)}{2} = |E| = m$$

$\square$

Proof #3 Let $P(n)$ denote that K_n has $\frac{n(n-1)}{2}$ edges.

We prove $\forall n \geq 1 : P(n)$ using induction over n.

Base case: $P(1)$ is true. That is, K_1 has $\frac{1(1-1)}{2} = 0$ edges. Yes, this is true!

Inductive case: We WTS $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$

Assume $P(n-1)$. That is, assume

$$K_{n-1} \text{ has } \frac{(n-1)(n-1-1)}{2} = \frac{(n-1)(n-2)}{2} \text{ edges.}$$

Now, consider an arbitrary clique K_n. Let K_n' be the graph created by removing one node and all its edges. Note that $K_n' = K_{n-1}$.

Goal: # edges of $K_n = \frac{n(n-1)}{2}$.

edges of K_n = # of edges K_{n-1} + # of edges we have to add to K_{n-1} to get K_n

$$= \frac{(n-1)(n-2)}{2} + n-1$$

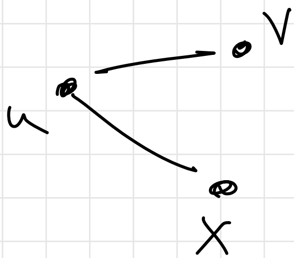
$$= \frac{n^2-3n+2}{2} \quad \frac{2(n-1)}{2}$$

$$= \frac{n^2 - 3n + \cancel{2} + 2n - \cancel{2}}{2}$$

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

We've proved the inductive case.

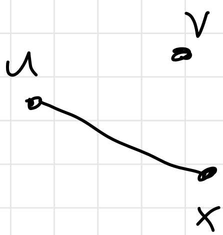
H



$V = \{u, v, x\}$

$E = \{\{u,v\}, \{u,x\}\}$

G by removing edge $\{u,v\}$



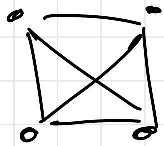
$\{\{u,v\}, \{u,x\}\} \setminus \{\{u,v\}\}$

$= \{\{u,x\}\}$ ✓

$\{\{u,v\}, \{u,x\}\} \setminus \{u,v\}$

$= \{\{u,v\}, \{u,x\}\}$ ✗

Last time: complete graphs / clique



real-world examples?

ex

there is a way to drive

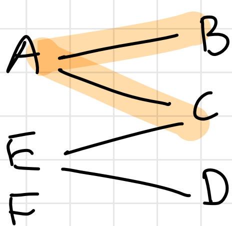
non-ex

direct interstate connections

ex a family,
edges = blood relations

Def A <u>bipartite graph</u> is a graph $G = (L \cup R, E)$ s.t. $L \cap R = \emptyset$ and $E \subseteq \{\{l, r\} : l \in L \wedge r \in R\}$.

ex



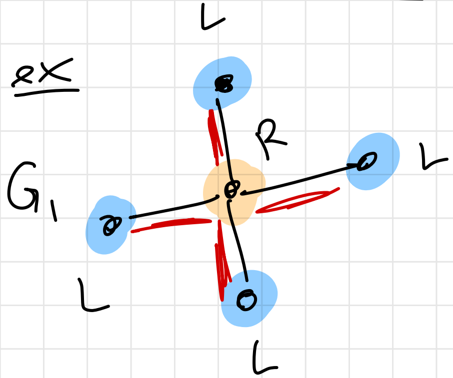
$V = \{A, B, C, D, E, F\}$

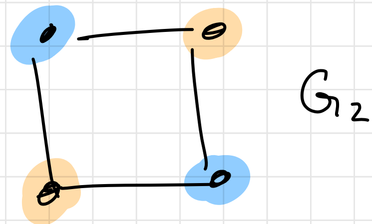
$L = \{A, E, F\}$
$R = \{B, C, D\}$

$L \cap R = \emptyset$ ✓

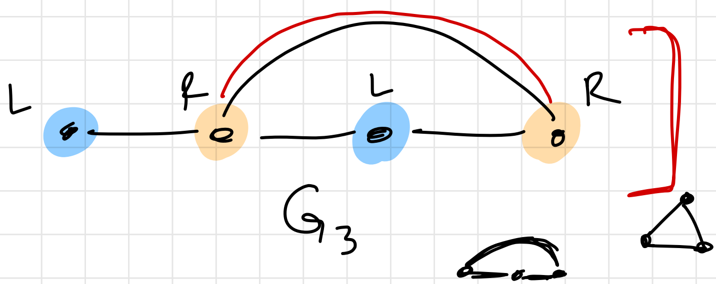
$E \subseteq \{\{l, r\} : l \in L \land r \in R\}$ ✓

ex



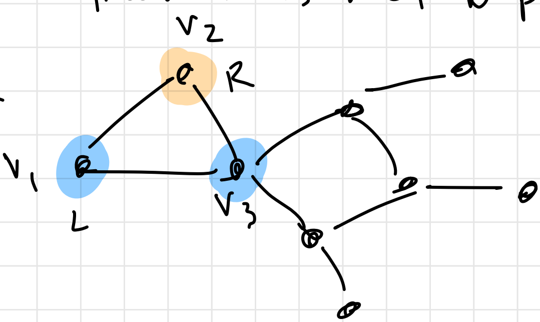


non-ex



Claim Let G be an undirected graph. If G contains a triangle, then it is not bipartite.

ex

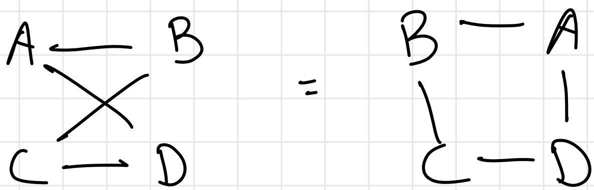


$\neg(p \Rightarrow q) \equiv p \land \neg q$

Proof Aiming for a contradiction, suppose that G contains a triangle and is bipartite.

Let v_1, v_2, v_3 be the nodes of the triangle. Without loss of generality, since we could relabel the nodes, suppose that $v_1 \in L$ and $v_2 \in R$. Since $v_2 \in R$, $v_3 \in L$. But there is an edge from v_1 to v_3 and both are in L, which contradicts that G is bipartite. $\square$

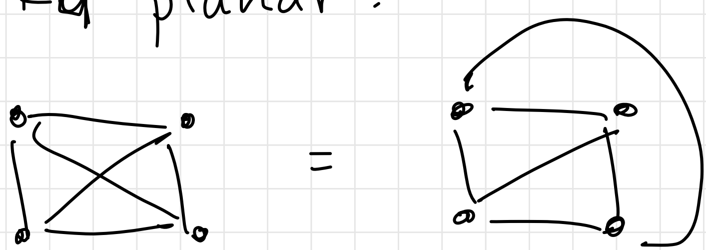
<u>Def</u> A graph is <u>planar</u> if we can draw it in the plane w/out edge crossings.



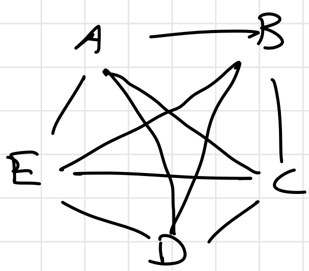
note: graphs are equal if verts equal and edges equal

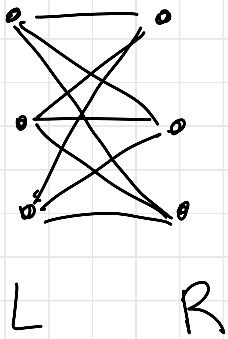
K_n is comple graph on n nodes

Is K_4 planar?



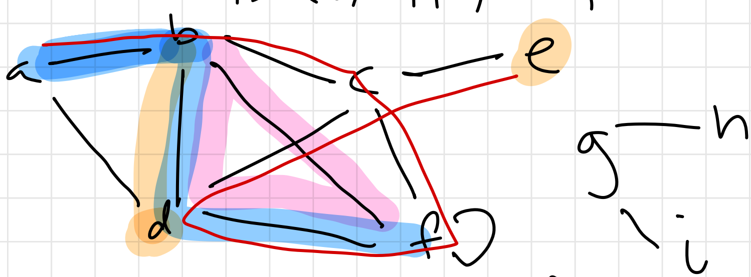
K_5 is not planar





complete bipartite graph

$K_{3,3}$ complete bipartite graph on 3 nodes is (b,d,f) a path?



is (e,d,b) a path? no!

(a,b,d,f) $(a,b,\underline{c},f,d,\underline{c},e)$

Def A path in $G=(V,E)$ is a sequence of nodes $(u_1,u_2,\dots,u_k)$ s.t.

① $\forall i \in \{1,2,\dots,k\} : u_i \in V$

$u_1 = a$
$u_2 = b$
$u_3 = d$
$u_4 = f$

- Is this def. done?
- Are there things that fit the def. but shouldn't be considered paths?
- How to fix?

② $\forall i \in \{1,2,\dots,k-1\} : (u_i, u_{i+1}) \in E$

(a,b,d,f) when $k=4$, $u_{k+1}=u_5$

(u_1, u_2)

$(a,b) \in E$

A path is simple if all its nodes are unique.

The length of a path is its # of edges.

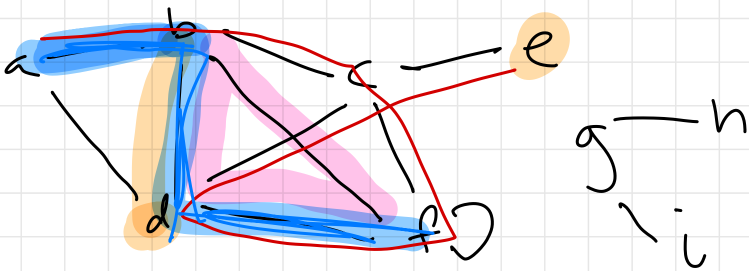
length of (a,b,d,f) is 3

in general, $k-1$.

The <u>shortest path</u> is the path of min. length between two nodes.

The <u>distance</u> $dist(u,v)$ or $d(u,v)$ between u,v is the length of the shortest path between u,v.

$d(a,f)$? 2, (a,b,f).



A graph is <u>connected</u> if $\forall u,v \in V$, $\exists$ a path from u to v.

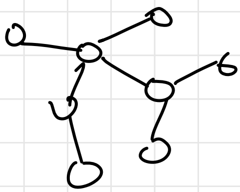
<u>Def</u> A <u>cycle</u> $(u_1, u_2, \ldots, u_k, u_1)$ is a path of length ≥ 2 from u_1 to u_1 that does not traverse the same edge twice.

A cycle is <u>simple</u> if its nodes are distinct.

A graph is <u>acyclic</u> if it contains no cycles.

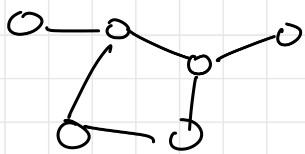
ex

acyclic



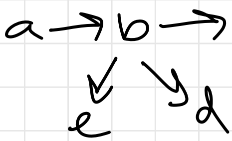
acyclic

non-ex

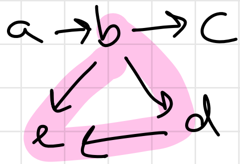


not acyclic

ex



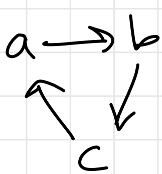
acyclic



acyclic

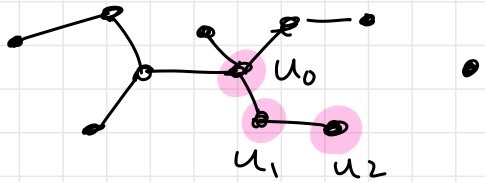
(b, d, e,

non-ex



(a, b, c, a)

Lemma 11.33 If $G = (V, E)$ is an undirected acyclic graph, then $\exists v \in V$ s.t. $\deg(v) = 0$ or $\deg(v) = 1$.



<u>Proof</u> We give a proof by construction via an algorithm that, given an undirected acyclic graph, finds a deg. 0 or deg. 1 node.

alg:

let u_0 be any node in V
let $i = 0$
while current node u_i has unvisited neighbors:
    let u_{i+1} = any such unvisited neighbor
    $i = i+1$
return u_i

Let t be the node returned by alg on G. WTS either $\deg(t) = 0$ or $\deg(t) = 1$.

Case 1: $t = u_0$. 

$\deg(t) = 0$.

Case 2: $t = u_k$, $k \geq 1$. We show $\deg(t) = 1$.

Since t is last in $(u_0, u_1, \ldots, u_k)$, there is no edge from t to any unvisited node. If $\exists$ edge from t to any other node u_j other than u_{k-1}, it is in $(u_0, u_1, \ldots, u_{k-2})$

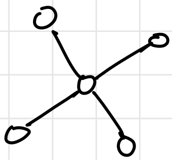
$$u_0 - u_1 - u_2 - \cdots\ u_j\ \cdots - u_{k-1} - \overset{=t}{u_k}$$

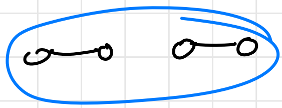
But then $(u_j, \ldots, u_{k-1}, u_k, u_j)$ is a cycle. So no such edge exists, and t has only one edge back to u_{k-1}. So $\deg(t) = 1$.

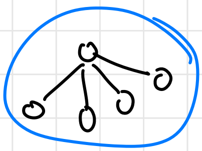
Def A *tree* is an undirected graph that is connected and acyclic.

ex

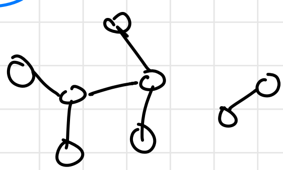
5 verts
4 edges







non-ex



(A forest)

Thm (chapter 11)

If $T = (V, E)$ is a tree, then

$$|E| = |V| - 1$$

Thm If $T = (V, E)$ is a tree, then

① Adding an edge creates a cycle

② Removing an edge disconnects the

graph.