In Computer Science, recursion is a common strategy for solving problems.

- take a problem instance
- split it into subproblems...
- ... until they are small

<u>ex</u> binary search

Problem: find an element in a sorted array.

$$A = (\underbrace{a_1, a_2, a_3}, \underbrace{\ldots, a_n})$$

$$\begin{tikzcd}
 & & & A_l \arrow[dl] \arrow[dr] & & & A_r \arrow[dl] \arrow[dr] & \\
 & & A_{l_l} & & A_{l_r} & A_{r_l} & & A_{r_r}
\end{tikzcd}$$

$$\vdots$$

base case: single element array.

Mathematical Induction is a proof technique that is analogous to recursion.

ex to prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$,

$$P(n) = \begin{cases} 1+2+3+\cdots+n = \frac{n(n+1)}{2} & : T \\ \neq & : F \end{cases}$$

We prove that the formula holds for $n=0$ (base case) and that if it holds for $n \geq 1$, then it holds for $n+1$.

^ some specific

Let P be a predicate concerning ints ≥ 0. To give a proof by mathematical induction that $\forall n \geq 0 : P(n)$, we prove 2 things:

$\forall n \geq 0 \; P(n) \Rightarrow P(n+1)$

(1) Base case: $P(0)$

(2) Inductive case: $\forall n \geq 1$, prove that $P(n-1) \Rightarrow P(n)$

If we do (1) and (2), we've proved $\forall n \geq 0 : P(n)$.

why?

~~(5.1 in book)~~

ex Suppose we have proven $P(0)$ and $P(n-1) \Rightarrow P(n)$. These establish

$P(3)$.

Proof WTS $P(3)$.

statement	reason
$P(0)$	assumption
$P(0) \Rightarrow P(1)$	plug in $n=1$ to $P(n-1) \Rightarrow P(n)$, assumption
$P(1)$	because $P(0) \Rightarrow P(1)$, and we have $P(0)$ (modus ponens)
$P(1) \Rightarrow P(2)$	plug in $n=2$ to $P(n-1) \Rightarrow P(n)$
$P(2)$	modus ponens
$P(2) \Rightarrow P(3)$	plug in $n=3$ to $P(n-1) \Rightarrow P(n)$
$P(3)$	modus ponens

Claim $\underbrace{\forall n \geq 0, \quad \sum_{i=0}^{n} 2^i = 2^{n+1} - 1}$

$n \in \mathbb{Z}$

$$\underbrace{2^0 + 2^1 + 2^2 + \cdots + 2^n}_{LHS} = \underbrace{2^{n+1} - 1}_{RHS}$$

ex	n	LHS	RHS
$\longrightarrow$	0	$2^0 = 1$	$2^1 - 1 = 2 - 1 = 1$ ✓
	1	$2^0 + 2^1 = 1 + 2 = 3$	$2^{1+1} - 1 = 2^2 - 1 = 4 - 1 = 3$ ✓
	2	$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$	$2^3 - 1 = 8 - 1 = 7$ ✓

Steps to prove a "$\forall n \geq 0$: ______" statement using mathematical induction:

① Clearly state $P(n)$ and that your proof is by mathematical induction. And state the variable you are performing induction over.

② Prove $P(0)$ (base case)

③ Prove $\forall n \geq 1: P(n-1) \Rightarrow P(n)$ (inductive case)

Claim: $\forall n \geq 0: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Proof:

① We define $P(n)$ to mean that
$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

$$P(n) = \begin{cases} T & \text{if } \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \\ F & \text{otherwise} \end{cases}$$

We show by induction that $\forall n \geq 0 : P(n)$.
We use induction over n.

② For the base case, we WTS $P(0)$.

That is, WTS $\sum_{i=0}^{0} 2^i = 2^0 = 2^{0+1} - 1$ $\uparrow$ $n=0$

$1 = 2^1 - 1 = 2 - 1 = 1$ ✓

LHS = RHS.

③ For the inductive case, we need to prove $\forall n \geq 1 : P(n-1) \Rightarrow P(n)$.

That is, $\forall n \geq 1$:

$$\boxed{\sum_{i=0}^{n-1} 2^i = 2^{(n-1)+1} - 1} \Rightarrow \underbrace{\sum_{i=0}^{n} 2^i}_{\text{LHS}} = \underbrace{2^{n+1} - 1}_{\text{RHS}}.$$

$\leftarrow$ inductive hypothesis

Assume $P(n-1)$.

WTS $P(n)$.

$$\text{LHS} = \sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} 2^i + 2^n$$ def. of summations

$$= 2^{(n-1)+1} - 1 + 2^n$$ subs. into inductive hyp.

$$= 2^n - 1 + 2^n$$

$$= 2^{n+1} - 1$$ algebra

$= RHS$

So we have shown $P(n)$.

We've shown $P(0)$ and $P(n-1) \Rightarrow P(n)$, so by the principle of mathematical induction, $P(n)$ holds $\forall n \geq 0$. $\square$

Now - 4:53 : worksheet → turn in w/ your name

4:55 - end : do together

Recall the steps for proving a statement "$\forall n \geq 0$: something" using mathematical induction:

(1) Clearly state the property $P(n)$, that you are using mathematical induction, and what variable you are doing induction over.

(2) Prove the base case: $P(0)$.

(3) Prove the inductive case: $P(n-1) \Rightarrow P(n)$.

In this activity, you will prove that $\forall n \geq 0$

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Answer the following questions:

- Do you believe the claim? I give you an example of it holding below. Give at least two more examples of n for which the claim holds by filling in two more rows of the table for different n.

n	$\sum_{i=0}^{n} i$	$\frac{n(n+1)}{2}$
1	$\sum_{i=0}^{1} i = 0 + 1 = 1$	$\frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

- What is the predicate $P(n)$ that you will prove holds $\forall n \geq 0$?

$P(n)$ is $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- What variable will you do induction over?

n

- What is the base case? (Don't just write $P(0)$; translate it into the specific $P(n)$ you defined above.)

$\sum_{i=0}^{0} i = \frac{0(0+1)}{2}$

- What is the inductive case? (Again, don't just write $P(n-1) \Rightarrow P(n)$; translate it into the specific $P(n)$ you defined above.)

$\forall n \geq 1: \boxed{\sum_{i=0}^{n-1} i = \frac{(n-1)(n-1+1)}{2}} \Rightarrow \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- What is the inductive hypothesis?

Now that you have answered the above questions, you are ready to write the full proof! The three steps are labeled in the proof for you to fill in.

Proof.

(1) Let $P(n)$ be $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

We show that $\forall n \geq 0$: $P(n)$ using induction over n.

(2) For the base case, we prove that $P(0)$.

Let $n=0$. Then $\sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0$. Also, $\frac{n(n+1)}{2} = \frac{0(0+1)}{2} = 0$.

So $P(0)$ holds.

(3) For the inductive case, we prove that $\forall n \geq 1$, $P(n-1) \Rightarrow P(n)$.

Assume $n \geq 1$ and $P(n-1)$. That is, we assume $\boxed{\sum_{i=0}^{n-1} i} = \frac{(n-1)(n-1+1)}{2} = \boxed{\frac{(n-1)(n)}{2}}$.

We WTS $P(n)$; that is, that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

LHS RHS

LHS

$\boxed{\sum_{i=0}^{n} i} = \boxed{\sum_{i=0}^{n-1}} + n$ by def. of Σ

$= \frac{(n-1)n}{2} + n$ subs. w/ inductive hypothesis

$= \frac{n^2-n}{2} + \frac{2n}{2}$

$= \frac{n^2-n+2n}{2}$ algebra

RHS

$= \frac{n^2+n}{2} = \boxed{\frac{n(n+1)}{2}}$, so $P(n)$ holds. □

Because we showed $P(0)$ and $\forall n \geq 1: P(n-1) \Rightarrow P(n)$, by the principle of mathematical induction, $\forall n \geq 0: P(n)$. $\square$