Predicates

So far, our propositions:

$\underbrace{5 \text{ is even}}_{P}$

<u>Def</u> A <u>predicate</u> is a Boolean-valued function $P: U \to \{T, F\}$ for a set U.

That is, a rule or property that a particular entity may or may not have.

<u>ex</u>

$$\text{isEven}(n) := \begin{cases} T & \text{if } n \text{ is even} \\ F & \text{if } n \text{ is odd} \end{cases}$$
$n \in \mathbb{Z}$

$$\text{isSubset}(A, B) := \begin{cases} T & \text{if } A \subseteq B \\ F & \text{if } A \not\subseteq B \end{cases}$$
A, B sets

$$\text{isRat}(x) := \begin{cases} T & \text{if } x \in \mathbb{Q} \\ F & \text{if } x \notin \mathbb{Q} \end{cases}$$
$x \in \mathbb{R}$

On its own, a predicate $P(x)$ has

no truth value. The value x is unbound.

isEven(n)

$n \in \mathbb{Z}$

Can make a proposition by:

1) apply to specific entity

let $m = 5$.

isEven(m) F

isEven(6) T

2) Use quantifiers

For all integers n, isEven(n). F

T There exists n such that n is even

Universal quantifier $\forall$ "for all"

$\forall n \in \mathbb{Z}$: isEven(n)

$\forall x \in S : P(x)$

"for all x in S, P(x) is true"

T if P(x) evaluates to T for all $x \in S$.

Existential quantifier $\exists$ "there exists"

$\exists x \in S : P(x)$

"there exists an x in S such that P(x) is true"

T if P(x) evaluates to T for some $x \in S$.

Examples of propositions w/ quantifiers:

$\forall n \in \mathbb{Z} : \text{isEven}(2n)$ T

$\forall n \in \mathbb{Z} : \text{isEven}(n^2) \Rightarrow \text{isEven}(n)$ T

$\exists n \in \mathbb{Z} : \text{isPrime}(n)$ T

$\forall n \in \mathbb{Z} : \text{isPrime}(n)$ F

rationals

$\exists x, y \in \mathbb{R} : xy \in \mathbb{Q} \wedge \neg(x \in \mathbb{Q} \wedge y \in \mathbb{Q})$

reals, not, and

"there exist real numbers x and y such that x times y is rational and not both x and y are rational"

T

how to show $\exists$ is true? give one!

$x = \sqrt{2}$, $y = \sqrt{2}$. $xy = \sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2$

<u>Precedence rules:</u>

- $\forall$, $\exists$ have highest precedence
- () to override or for clarity

<u>ex</u>

$$\forall x \in S : P(x) \Rightarrow \exists y \in S : P(y)$$

$$\left(\forall x \in S : P(x)\right) \Rightarrow \left(\exists y \in S : P(y)\right)$$

$$\forall x \in S : \left(P(x) \Rightarrow \exists y \in S : P(y)\right)$$

All students do not pay full tuition.

Let S be the set of all students.

Let $P(x)$ be "x pays full tuition"

$$\forall x \in S : \neg P(x)$$

no student pays full tuition.

$$\exists x \in S : \neg P(x)$$

$$=$$

$$\neg(\forall x \in S : P(x))$$

Not all students pay full tuition

If n^2 is even, then n is even.

$$\forall n \in \mathbb{Z} : \text{isEven}(n^2) \Rightarrow \text{isEven}(n)$$

Predicates

$P(x)$ $\quad x \in U$ $\quad$ isEven(x)

$\downarrow \quad \downarrow$ $\quad\quad$ $x \in \mathbb{Z}$

T $\quad$ F

```
if x % 2 == 0:
    do something
while (x > 2) {
    do something
```

greaterThan2(x)

$x \in \mathbb{R}$

Quantifiers

$\forall$ "for all" universal quantifier

$\forall x \in S : P(x)$

"for all x in S, P(x) is T"

$\exists$ "there exists" existential quantifier

$\exists x \in S : P(x)$

"there exists x in S such that P(x) is T" checks $\forall x \in S: P(x)$

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for x in S:
    if not P(x) then
        return False
return True
```

checks $\exists x \in S: P(x)$

```
for x in S:
    if Q(x) then
        return True
return False
```

<u>Def</u> A fully quantified expression is a <u>theorem</u> if it is T for every possible meaning of its predicates and sets.

$\forall S$

<u>Thm (3.39)</u> Let S be any set.

$$\forall x \in S: [P(x) \vee \neg P(x)] \qquad \forall P$$

<u>ex</u> $P(x) = \text{isEven}(x)$, $S = \mathbb{Z}$

$$\forall x \in \mathbb{Z}: [\text{isEven}(x) \vee \neg\text{isEven}(x)]$$

Non-theorem (3.40)

$$\overset{F}{[\forall x \in S : P(x)]} \vee \overset{F}{[\forall x \in S : \neg P(x)]}$$

note: implied $\forall S$, $\forall P$

Disproof that ↑ is a theorem by counterexample.

$S = \{1, 2\}$

$P = \text{isEven}(x)$

$$\overset{F}{[\forall x \in \{1,2\} : \text{isEven}(x)]} \vee \overset{F}{[\forall x \in \{1,2\} : \neg \text{isEven}(x)]}$$

↓ consider $x = 1$

↓ Consider $x = 2$

<u>Def</u> Fully quantified expressions φ, ψ are logically equivalent ($\varphi \equiv \psi$, $\varphi \Leftrightarrow \psi$) if "$\varphi \Leftrightarrow \psi$" is a theorem.

Theorem (3.41)

$$\underbrace{\neg\left[\forall x \in S : P(x)\right]}_{\varphi} \iff \underbrace{\left[\exists x \in S : \neg P(x)\right]}_{\psi}$$

$$\varphi \equiv \psi$$

Theorem (3.42)

$$\neg\left[\exists x \in S : Q(x)\right] \iff \left[\forall x \in S : \neg Q(x)\right]$$

Theorem $\forall x \in \emptyset : P(x)$ "$P(x)$ is vacuous-ly true"