# Relations

CS application: relational databases

| student id | first name | last name |
| --- | --- | --- |
| 123 | Bob | Smith |

| student id | passed course |
| --- | --- |
| 123 | CSCI 246 |
| 123 | CSCI 127 |

Questions about data stored in relational databases can be posed precisely using the language of relations.

SQL (structured query language)

**Def** The cartesian product of two sets $A, B$ is

$$A \times B = \{ (a, b) : a \in A \wedge b \in B \}$$

lists/tuples/arrays — order matters

ex

$\mathbb{R} \times \mathbb{R}$ = 2d plane, Cartesian plane

$\{red, blue\} \times \{1,2,3\} = \{(red,1), (red,2), (red,3), (blue,1), (blue,2), (blue,3)\}$

Q what is $|A \times B|$? $|A|, |B|$

$|A| \cdot |B|$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$

Def A binary relation $R$ on sets $A, B$ is a subset $R \subseteq A \times B$.

We write $(x,y) \in R$ as $xRy$

$(x,y) \notin R$ as $x \not R y$

examples

1) $R_1$ "is (blood) related to" is a binary relation on people.

let $P$ be the set of all people

"is blood related to" is

$\{(x,y) : x \in P \wedge y \in P \wedge x \text{ is related to } y\}$

(Serena Williams, Venus Williams) $\in R$,

(Lucy Williams, Serena Williams) $\notin R$,

② $<$ on $A = \{1, 2, 3, 4\}$

$< \; = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

$1 < 2$ but $3 \not< 2$

③ Let $f: A \to B$ be a function

$\{(a, f(a)) : a \in A\} \subseteq A \times B$, so it is a relation

Q Is the converse true? Let $R$ be a binary relation on $A, B$.

$\{(x,y) : x \in A \wedge y \in B \wedge xRy\}$

$\Rightarrow f: A \to B$ s.t. $f(x) = y$ is a function

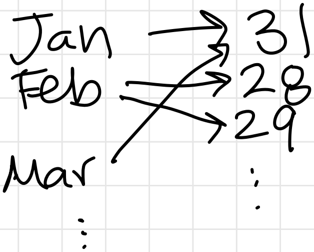
true or false

(4) Let $A$ = months, $B$ = number of days

Relation: month, its # days

$\{(\text{Jan}, 31), (\text{Feb}, 28), (\text{Feb}, 29), (\text{Mar}, 31) \ldots \}$

| | |
|---|---|
| Jan | 31 |
| Feb | 28 |
| Feb | 29 |
| Mar | 31 |
| ⋮ | ⋮ |



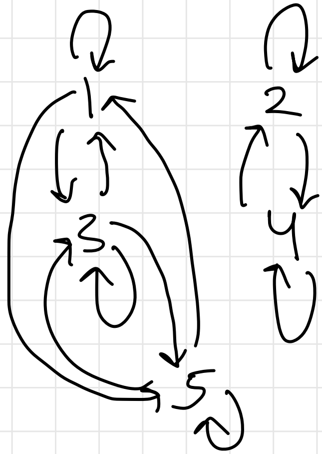

(5) $A = \{1, 2, 3, 4, 5\}$

$(1,1) \in R_2$

$(2,4) \in R_2$

$(3,2) \notin R_2$

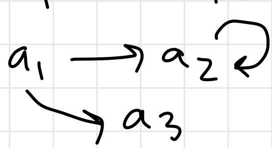
$R_2$:




## Properties of relations

Let $R \subseteq A \times A$, so $R$ is a relation on $A$

$R$:



$R$ is <u>reflexive</u> if $\forall a \in A: aRa$

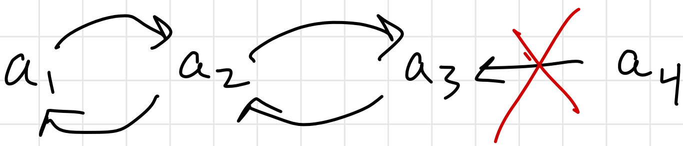
all nodes have self-loops

$R$ is <u>irreflexive</u> if $\forall a \in A: a \not R a$

no nodes have self-loops

$R$ is <u>symmetric</u> if

$$\forall a_1, a_2 \in A: \ a_1 R a_2 \Rightarrow a_2 R a_1$$

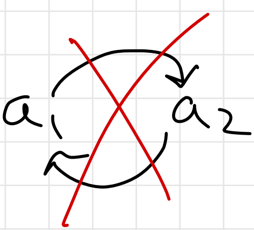


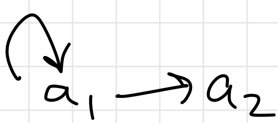
whenever we have a forward edge, we have the backward edge.

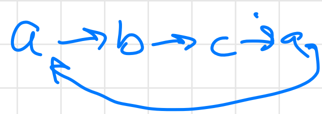
$\{(a,b), (b,c), (c,a$

$R$ is <u>anti-symmetric</u> if

$$\forall a_1, a_2 \in A: (a_1 R a_2 \wedge a_2 R a_1) \Rightarrow a_1 = a_2$$







never have backwards edges, but self-loops okay.

R is <u>transitive</u> if

$$\forall \underline{a}, \underline{b}, \underline{c} \in A : \overset{T}{\underset{a_1 R a_2 \wedge a_2 R a_1}{(\underline{a} R \underline{b} \wedge \underline{b} R \underline{c})}} \Rightarrow \overset{F}{\underset{a_1 \not R a_1}{(\underline{a} R \underline{c})}}$$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

shortcut edges always exist

<u>Q</u> Is $\begin{tikzcd} a_1 \arrow[r, bend left] & a_2 \arrow[l, bend left] \end{tikzcd}$ transitive? $a_1 \neq a_2$

let $\underline{a = a_1}$

$b = a_2$

$\underline{c = a_1}$

$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[l, bend left] \end{tikzcd}$$

<u>Q</u> Is $\begin{tikzcd} a_1 \arrow[loop above] \end{tikzcd}$ transitive?

$$\overset{T}{(a_1 R a_1 \wedge a_1 R a_1)} \Rightarrow \overset{T}{(a_1 R a_1)}$$

# Relations review

$(a,b) \in R$

$aRb$

Let $A, B$ be sets.

$R \subseteq A \times B$ is a binary relation

often, we are concerned with relations over a single set:

$R \subseteq S \times S$ "$R$ is a relation on $S$"

Properties of relations on single sets:

- reflexive: $\forall a \in A : aRa$
- irreflexive: $\forall a \in A : a \not R a$
- symmetric: $\forall a_1, a_2 \in A : a_1 R a_2 \Rightarrow a_2 R a_1$

  $a_1 \rightleftarrows a_2$
- anti-symmetric: $\forall a_1, a_2 \in A :$

  $a_1 \quad a_2 \quad \underbrace{(a_1 R a_2 \wedge a_2 R a_1)} \Rightarrow a_1 = a_2$
- transitive:

  $\forall a_1, a_2, a_3 \in A : (a_1 R a_2 \wedge a_2 R a_3) \Rightarrow a_1 R a_3$

symmetric ✓
anti-symmetric ✓

a b

$A = \{a, b\}$ $R \subseteq A \times A$

$\emptyset$

$B \subseteq People \times People$

(Lucy, Britney Spears) $\in B$

Lucy $B$ Britney Spears

Lucy $\not B$ Braeden

$> \subseteq \mathbb{R} \times \mathbb{R}$

$2 > 1.5$

$2 \not> 2$

ex relation $<$ on $\mathbb{Z}$:

- reflexive?

no - disproof by counterexample.

$1 \in \mathbb{Z}$. $1 \not< 1$

- irreflexive? yes.

$\forall a \in \mathbb{Z}: a \not R a$.

let $a \in \mathbb{Z}$. $a \not< a$ because no integer is less than itself. ∎

- symmetric?

disproof by counterexample:

$1 < 2$ but $2 \not< 1$.

- antisymmetric? $\forall a_1, a_2 \in A$:
$(a_1 R a_2 \wedge a_2 R a_1) \Rightarrow a_1 = a_2$

Let $a_1, a_2 \in \mathbb{Z}$. Assume $a_1 < a_2$ and $a_2 < a_1$.

Since no $a_1, a_2$ satisfy $a_1 < a_2$ and $a_2 < a_1$,

$(a_1 < a_2 \wedge a_2 < a_1) \Rightarrow a_1 = a_2$ is

vacuously true.

- tranisitive?

$\forall a_1, a_2, a_3 \in A: (a_1 R a_2 \wedge a_2 R a_3) \Rightarrow a_1 R a_3$

Proof Assume $a_1, a_2, a_3 \in \mathbb{Z}$ and $a_1 < a_2$ and $a_2 < a_3$. By the def. of $<$, $a_1 < a_3$

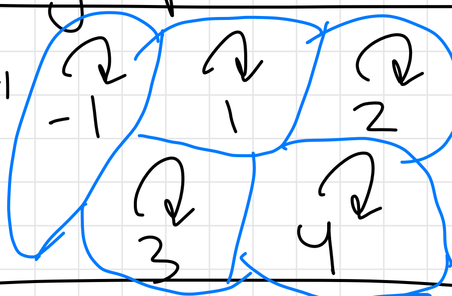
**Def** A binary relation $R$ is an *equivalence relation* if it is reflexive, symmetric, and transitive.

$a \rightleftarrows b$ $\qquad$ $a \to b \to c$, $a \to c$

Consider $A = \{-1, 1, 2, 3, 4\}$

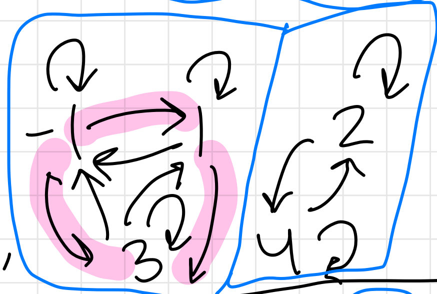
graph

$(-1,-1) \in R_1$
$(1,-1) \notin R_1$
$=$



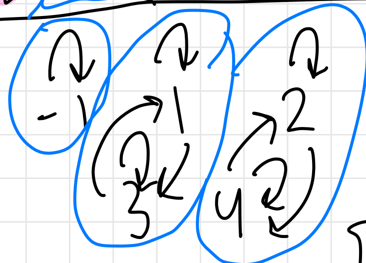

$[-1] = \{-1\}$
$[1] = \{1\}$
$[2] = \{2\}$
$\vdots$

same parity (evenness, oddness)




$[-1] = \{-1, 1, 3\}$
$= [1] = [3]$
$[2] = \{2, 4\} = [4]$

same sign (+/-) and same parity




$[-1] = \{-1\}$
$[1] = \{1, 3\} = [3]$
$[2] = \{2, 4\} = [4]$

**Def** For an equivalence relation $R$ on set $A$, the <u>equivalence class</u> of $a \in A$ is

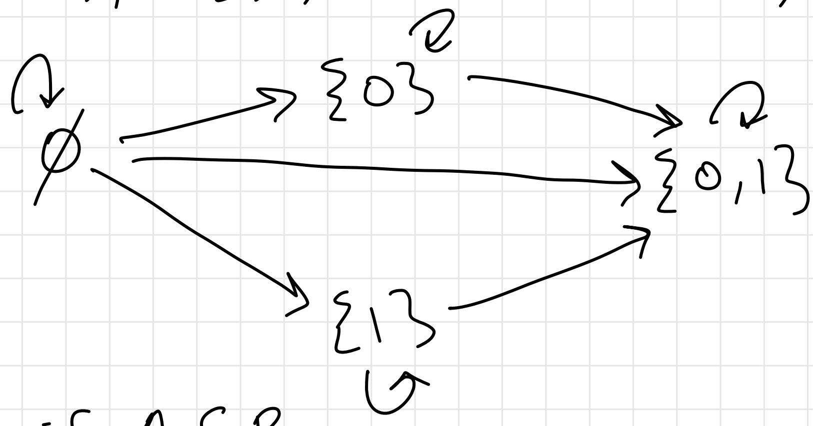
$$[a] \text{ is } \{x \in A : x R a\}$$

"the equivalence class of a"

Another relation:

$\subseteq$ on $\mathcal{P}(\{0,1\})$

$$\mathcal{P}(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$$




$A R B$ if $A \subseteq B$

$\emptyset R \{0\}$ ?

$\{0,1\} R \{0,1\}$ $\quad \{0,1\} \subseteq \{0,1\}$

**Def** A binary relation $R$ on set $A$ is:

- a **partial order** if $R$ is
  - reflexive,
  - transitive,
  - anti-symmetric

- a **strict partial order** if $R$ is
  - irreflexive,
  - transitive,
  - anti-symmetric

**Def** A partial order is **total order** if all pairs of different elements from $A$ are comparable.

↳ $\forall a, b \in A : (a \neq b) \Rightarrow (aRb \vee bRa)$

A strict partial order is a **strict total order** if all pairs of diff. elts. are comparable.

| | $< \subseteq \mathbb{R}\times\mathbb{R}$ | Same Bday $\subseteq P\times P$ | Subset $\subseteq P(S)\times P(S)$ |
|---|---|---|---|
| reflexive | N | Y | Y |
| irreflexive | Y | N | N |
| symmetric $1 \rightleftarrows 2$ | N | Y | N |
| anti-symmetric | Y | N | Y |
| transitive $1 \to 5 \to 10$ | Y | Y | Y |
| equivalence relation | N | Y | N |
| partial order | N | N | Y |
| strict partial order | Y | N | N |
| total order | N | N | N |
| strict total order | Y | N | N |

$$\{0\} \rightarrow \{0,1\}, \qquad \{0\} \not\leftarrow \{0,1\}$$