Def A *set* is an unordered collection of distinct items called *elements*.

ex $D = \{0, 1, 2, 3, \ldots, 9\}$ has 10 elements

bits $= \{0, 1\}$ has 2 elements

$\mathbb{Z}$ = set of all integers

$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

has infinite elements

$\mathbb{Q}$ = rationals

$\mathbb{R}$ = reals

$V = \{a, e, i, o, u, y\}$ has 6 elts

$A = \{-20, \pi, a\}$

Def Two sets A, B are *equal* $(A = B)$ if A and B contain exactly the same elements.

ex. $\{0, 1\} = \{1, 0\}$

Def We write $x \in S$ if x in S.

"x is an element of S"

We write $x \notin S$ if x not in S.

ex $0 \in \text{bits} = \{0, 1\}$

$2 \notin \text{bits}$

$\pi \notin \mathbb{Z}$

Def The cardinality or size of set S is the number of distinct elements of S.

$|S|$

ex $|\text{bits}| = 2$

$|\{\underbrace{\{3,4\}}, \underbrace{\text{cat}}\}| = 2$

Q Can we have a set such that (s.t.) $|S| = 0$?

Def The empty set, denoted $\{\}$ or $\emptyset$, is the set with no elements.

$|\emptyset| = 0$

$|\{\emptyset\}| = 1$

$F = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

$|F| = 3$

Q If $A = B$, does $|A| = |B|$? T

Is the converse true? 2 min on own

if $|A| = |B|$, then $A = B$. F

Pf by counter example:

$A = \{0\}$ $B = \{1\}$

$|A| = 1$, $|B| = 1$, but $A \neq B$.

<u>Def</u> Set builder notation defines a set

$$S = \{x : \text{a rule about } x\}$$

such that

S contains the elements x s.t. the rule about x is true.

<u>ex</u> evens $= \{x : x \in \mathbb{Z} \text{ and } x \text{ even}\}$

"x such that x is in integers and x even"

$$\begin{cases} \text{evens} = \{x : x = 2c \text{ for } c \in \mathbb{Z}\} \\ \text{evens} = \{x : x \in \mathbb{Z} \text{ and } 2 \mid x\} \end{cases}$$

"2 divides x"

<u>Def</u> A is a subset of B (denoted $A \subseteq B$) if every element of A is also in B.

<u>ex</u> evens $\subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Q: $\mathbb{R} \subseteq \mathbb{Q}$? no, $\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$.

Q: $\emptyset \subseteq \mathbb{R}$ T

Note: $\emptyset \subseteq S$ for all sets S.

$S \subseteq S$ for all sets S.

($A \subset B$ means A is a strict subset of B, $A \subseteq B$ and $|A| < |B|$)

Note: if $A \subseteq B$, then $|A| \leq |B|$.

Q: Is the converse true?

If $|A| \leq |B|$, then $A \subseteq B$ F

divides

claim $\{x \in \mathbb{Z} : 18 \mid x\} \subseteq \{x \in \mathbb{Z} : 6 \mid x\}$

Step 1: understand claim!

The numbers divisible by 18 are contained/ a part of the numbers divisible by 6.

Every number divisible by 18 is also divisible by 6.

Step 2: do some examples.

ex x	$18 \mid x$?	$6 \mid x$?
18	T	T
4	F	F
	T	F

Pf Want to show $\{x \in \mathbb{Z} : 18 | x\} \subseteq \{x \in \mathbb{Z} : 6 | x\}$
which is to say, if $a \in \{x \in \mathbb{Z} : 18 | x\}$,
then $a \in \{x \in \mathbb{Z} : 6 | x\}$ by def. of $\subseteq$.
assume $a \in \{x \in \mathbb{Z} : 18 | x\}$.

$a = 18c$ for $c \in \mathbb{Z}$	def. of divisibility by 18
$a = 6 \cdot 3 \cdot c$	factoring
$a = 6 \cdot k$ for $k \in \mathbb{Z}$	because 6c is an integer, because int · int = int
$6 \| a$	def. of divisibility

$a \in \{x \in \mathbb{Z} : 6 | x\}$

$\square$

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Sets review

recall $\mathbb{Z}$ = set of all integers = $\{\ldots,-2,-1,0,1,\ldots\}$

$2 \in \mathbb{Z}$ $\qquad$ $1.5 \notin \mathbb{Z}$

$\{2,4\} \subseteq \mathbb{Z}$ $\qquad$ $\underbrace{\{x \in \mathbb{Z} : 2 \mid x\}}_{\text{evens}} \subseteq \mathbb{Z}$

is $2 \subseteq \mathbb{Z}$ ← F

"2 subset of the integers"

$\{2\} \subseteq \mathbb{Z}$ $\qquad$ T

"the set containing 2 is a subset of the integers"

Def $A \cup B$ "A union B" is $\{x : x \in A \text{ or } x \in B\}$

note that elements $x \in A$ and $x \in B$ are in $A \cup B$.

ex $\{2,4,6\} \cup \{2,3,4\} = \{2,3,4,6\}$

evens $\cup$ odds $= \mathbb{Z}$

$$\mathbb{R}^{\geq 0} \cup \mathbb{R}^{\leq 0} = \mathbb{R}$$

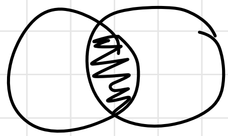
reals ≥ 0 reals ≤ 0

$A \cup \emptyset = A$

$A \cup A = A$ for all sets A

$\cancel{2 \cup \{1,3\} =}$ $\{2\} \cup \{1,3\} = \{1,2,3\}$

Def $A \cap B$ "A intersect B" $\{x : x \in A \text{ and } x \in B\}$

A  B

$B \cap A = A \cap B$

not disjoint

$\{2,4,6\} \cap \{2,3,4\} = \{2,4\}$ evens   odds

evens $\cap$ odds $= \emptyset$ disjoint

$A \cap \emptyset = \emptyset$ disjoint

for all sets A

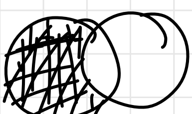
$A \cap A = A$

$\mathbb{R}^{\geq 0} \cap \mathbb{R}^{\leq 0} = \{0\}$

Def Sets A, B are disjoint if $A \cap B = \emptyset$.

Are $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^{\leq 0}$ disjoint? no

Def $A - B$ or $A \setminus B$ "A minus B" $\{x : x \in A$ and $x \notin B\}$

A  B ex $\{2,4,6\} - \{2,3,4\} = \{6\}$

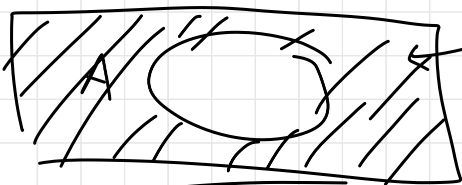
$\{2,3,4\} - \{2,4,6\} = \{3\}$

evens − odds = evens

$A - B \subseteq A$

$A - \emptyset = A$ ~~compliment~~

Def $\bar{A}$ or $\sim A$ "A complement" $\{x : x \notin A\}$



Universe

U

ex $\overline{\{2,4,6\}} = \{0,1,3,5,7,8,9\}$

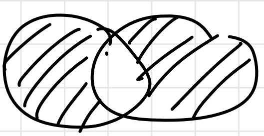
if U is $\{x \in \mathbb{Z} : 0 \le x \le 9\}$

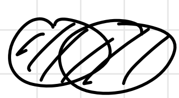
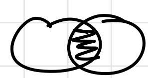
$\overline{\{2,4,6\}} = \{\ldots,-2,-1,0,1,3,5,7,8,9,10,\ldots\}$

if U is $\mathbb{Z}$

Def $A \oplus B$ "A exclusive or"

$(A \cup B) - (A \cap B)$

A  B

$0 = 2k$

for $k \in \mathbb{Z}$

$0 = 2(0)$

2 divides x

↓

9 divides x

↓

claim $\overset{A}{\{x\in\mathbb{Z} : 2|x\}} \cap \overset{B}{\{x\in\mathbb{Z} : 9|x\}}$
$\subseteq \underset{C}{\{x\in\mathbb{Z} : 6|x\}}$

if a number is divisible by 2 and 9, then it is divisible by 6.

$A\cap B \subseteq C$ if $x\in A\cap B$, then $x\in C$.

$P\subseteq Q$ if $y\in P$ then $y\in Q$

examples

x	$x\in A\cap B$	$x\in C$
6	F $x\notin B$	T
0	T	T

$\underbrace{\text{T} \qquad\qquad\qquad \text{F}}$ ← what a counter example would look like

Proof   Assume $x\in A\cap B$. Want to show $x\in C$.

statement	reason
$x\in A$ and $x\in B$	def. of $\cap$
$2\|x$ and $9\|x$	def. of A, B

$x = 2c$ and $x = 9d$ for integers c, d — def. of divisibility

$\underbrace{2c}_{} = \underbrace{9d}_{}$ — substitution

$2 \mid 9d$ — def. of divisibility ($9d = 2k$ for $k \in \mathbb{Z}$)

$2 \mid d$ — because $2 \nmid 9$

does not divide

$d = 2y$ for $y \in \mathbb{Z}$ — def. of divisibility

$x = 9 \cdot 2 \cdot y$ — substitution

$x = 18e$ for $e \in \mathbb{Z}$

$x = 6 \cdot 3 \cdot e$ — factoring

$x = 6f$ for $f \in \mathbb{Z}$ — $f = 3 \cdot e \in \mathbb{Z}$ by int · int = int

$6 \mid x$ — by def. of divisibility

$x \in C$

9/8 on a paper w/ your name:

Write the set builder notation for:

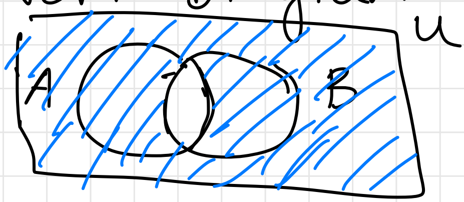
$A \cap B$ $\quad \{x : x \in A \text{ and } x \in B\}$

$A \cup B$ $\quad \{x : x \in A \text{ or } x \in B\}$

$\overline{A}$ $\quad \{x : x \notin A\}$

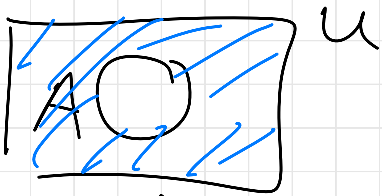
Draw the Venn Diagram for:

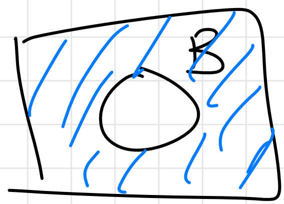
$\overline{A \cap B}$

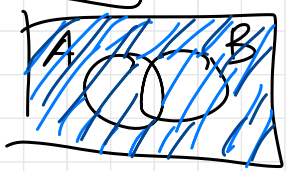


$=$

$(\overline{A} \cup \overline{B})$

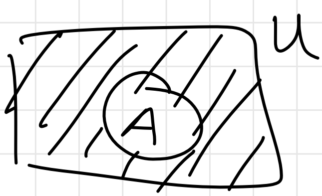






$(A \cap B) \cup C$

recall $\overline{A}$:



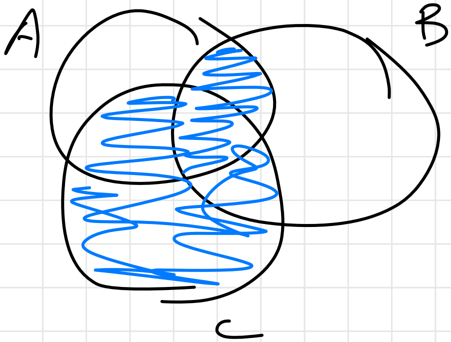
even integers

↓

recall set builder notation: $\{x \in \mathbb{Z} : 2 \mid x\}$

example

$(A \cap B) \cup C$



Def Given a set S, the power set of S is the set of all subsets of S.

$\mathcal{P}(S) = \{A : A \subseteq S\}$

$\emptyset \subseteq B$ for all sets B

ex $S = \{1, 2, 3\}$

$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\},$
$\{1,2\}, \{1,3\}, \{2,3\},$
$\{1,2,3\}\}$

$|\mathcal{P}(S)| = 8$ ✓

Fact $|\mathcal{P}(B)| = 2^{|B|}$ for all sets B.

ex $|S| = 3$, $2^3 = 2 \cdot 2 \cdot 2 = 8$ ✓

Note Power set is also denoted 2^B for set B.

$\mathcal{P}(B)$, 2^B same

$\emptyset \in \mathcal{P}(B)$ for all sets B

$B \in \mathcal{P}(B)$ for all sets B

Question: is $\emptyset \in \overline{\emptyset}$

Let's do an example. Suppose $U = \mathbb{Z}$.

$\overline{\emptyset} = \mathbb{Z}$ is $\emptyset \in \mathbb{Z}$? no.

$3 \in \mathbb{Z}$ $\{3\} \in \mathbb{Z}$

Question: is $\emptyset \subseteq \overline{\emptyset}$?

$\emptyset \subseteq S$ for all sets S

<u>Theorem</u> (De Morgan's Law)

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

<u>Proof</u> We prove the equivalent claim:

① $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ ②

Proof of ① Let $x \in \overline{A \cap B}$. WTS that $x \in \overline{A} \cup \overline{B}$.

Proof of 2 .

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__claim__ If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

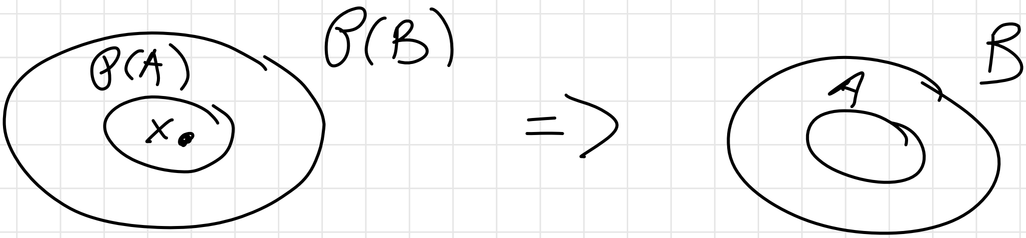
make sure we understand claim:

$\mathcal{P}(A)$ = set of all subsets of A

$= \{X : X \subseteq A\}$

$A \subseteq B$ = A is a subset of B

= if $x \in A$, then $x \in B$.



$A \subseteq B$

__examples__

$B = \{1, 2, \{1,3\}\}$ $\quad |\mathcal{P}(B)| = 2^{|B|} = 2^3 = 8$

$A = \{1\}$

$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{\{1,3\}\}, \{1, \{1,3\}\}, \{2, \{1,3\}\},$
$\{1, 2, \{1,3\}\}, \{1,2\}\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}\}$

$A \subseteq B$? yes. $\quad \mathcal{P}(A) \subseteq \mathcal{P}(B)$? yes

Let $A = \{5\}$, $B = \{6\}$.

Is $A \subseteq B$? no. $\mathcal{P}(A) \not\subseteq \mathcal{P}(B)$.

<u>Claim</u> If $\boxed{\mathcal{P}(A) \subseteq \mathcal{P}(B)}$ then $A \subseteq B$.

<u>Proof</u> Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. WTS $A \subseteq B$.

To show that $A \subseteq B$, we show that if $x \in A$, then $x \in B$.

Assume $x \in A$. WTS $x \in B$.

<u>statements</u>	<u>reasoning</u>
$\{x\} \subseteq A$	def of $\subseteq$
$\{x\} \in \mathcal{P}(A)$	def of $\mathcal{P}(A)$
$\{x\} \in \mathcal{P}(B)$	def of $\subseteq$, we know that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$
$x \in B$	def of $\mathcal{P}(B)$

Since whenever $x \in A$, $x \in B$, $A \subseteq B$. $\square$