Name ______________________________

Problem 1 (20 points (10 each))

The following are proposed proofs of the given claims. Below each one, write whether the proof is valid or not. If it is *not* valid, explain why. Only a sentence or so should be needed to explain why a proof does not work.

(a) *Claim.* Let x, y be real numbers. $|x| \cdot |y| = |x \cdot y|$.

Proof. We prove the claim using cases.

Case 1: $x \geq 0, y \geq 0$. Consider when $x = 5$ and $y = 10$. Then $|x| \cdot |y| = 5 \cdot 10 = 50$ and $|x \cdot y| = |5 \cdot 10| = |50| = 50$, so the claim holds.

Case 2: $x \leq 0, y \leq 0$. Consider when $x = -4$ and $y = -1$. Then $|x| \cdot |y| = 4 \cdot 1 = 4$ and $|x \cdot y| = |-4 \cdot -1| = |4| = 4$, so the claim holds.

Case 3: $x \geq 0, y \leq 0$. Consider when $x = 2$ and $y = -1.5$. Then $|x| \cdot |y| = 2 \cdot 1.5 = 3$ and $|x \cdot y| = |2 \cdot -1.5| = |-3| = 3$, so the claim holds.

Case 4: $x \leq 0, y \geq 0$. Consider when $x = -3$ and $y = \pi$. Then $|-3| \cdot |\pi| = 3 \cdot \pi = 3\pi$ and $|x \cdot y| = |-3 \cdot \pi| = |-3\pi| = 3\pi$, so the claim holds.

Since the cases are exhaustive and the claim holds in every case, the claim is true. □

(b) *Claim.* If $A \not\subseteq B$, then $A - B \neq \emptyset$.

Proof. Let $A - B \neq \emptyset$. We want to show that $A \not\subseteq B$.

statement	reasoning
$\exists a \in A - B$	given that $A - B \neq \emptyset$
$\exists a \in A : a \notin B$	definition of $-$
$\neg[\forall a \in A : a \in B]$	Theorem 3.41, which says that $\neg[\forall x : P(x)] \Leftrightarrow [\exists x : \neg P(x)]$
$\neg[A \subseteq B]$	definition of subset
$A \not\subseteq B$	definition of $\neg$

□

Problem 2 (30 points (10 each))

(a) Fill in the truth table for $p \Rightarrow q$.

p	q	$p \Rightarrow q$
T	T	
T	F	
F	T	
F	F	

(b) Construct a truth table for $(\neg(p \wedge (\neg q))) \Rightarrow (p \vee q)$. Show at least 3 columns of scratch work (your choice).

(c) Let $SortAlgs$ be the set of all sorting algorithms, $nArrays$ be the set of all n-element arrays, and $P(x, A)$ be a predicate that is true when algorithm A takes at least $n \log n$ steps on the the input x and false otherwise. Uses $SortAlgs$, $nArrays$, and $P(x, A)$ to write a fully-quantified statement of predicate logic to express the following:

"Every sorting algorithm takes at least $n \log n$ steps on some n-element input array."

Problem 3 (20 points (2 points each))

(a) True or false: for all sets S, $\emptyset \in S$.

(b) Let $S = \{1, 2, 3\}$. What is the power set of S?

(c) What is the converse of the statement "If n^2 is even, then n is even?"

(d) What is the contrapositive of the statement "If n^2 is even, then n is even?"

(e) Let $Q(x)$ be the predicate that says that x is prime. Is the statement $\forall x \in \mathbb{Z} : Q(x)$ true?

(f) Let $Q(x)$ be the predicate that says that x is prime. Is the statement $\exists x \in \mathbb{Z} : Q(x)$ true?

(g) Write an English sentence that is the logical negation of the sentence "Every entry in the array A is even."

(h) Let $L(x, y)$ be the predicate that says that x loves y. Write the fully quantified expression that corresponds to the statement that everyone loves themselves.

(i) Is the statement "All even prime numbers greater than 10 have a 5 as their last digit" true or false?

(j) Is the statement "In a group of 13 people, at least two share the same birth month" true or false?

Problem 4 (30 points)

Fill in the rest of the proof of the following claim.

Claim. $\{v \in \mathbb{Z} : 18|v\} \subseteq \{v \in \mathbb{Z} : 6|v\}$.

Proof. By the definition of subset, an equivalent statement is that if $x \in \{v \in \mathbb{Z} : 18|v\}$, then $x \in \{v \in \mathbb{Z} : 6|v\}$, so we prove that. Let $x \in \{v \in \mathbb{Z} : 18|v\}$. We want to show that $x \in \{v \in \mathbb{Z} : 6|v\}$.

statement	reasoning
$18\|x$	