$P(x)$ $\quad x \in \mathcal{U}$ $\quad$ isEven(x)

$\swarrow \quad \searrow$ $\qquad$ $x \in \mathbb{Z}$

$T \quad F$

Quantifiers

$\forall$ for all, universal quantifier

$\forall x \in S : P(x)$ "for all x in S, $P(x)$ is true"

T iff $P(x)$ is T for every $x \in S$

$\exists$ there exists, existential quantifier

$\exists x \in S : P(x)$ "there exists x in S s.t. $P(x)$ is true"

T iff $P(x)$ is T for some (≥ 1) $x \in S$.

<u>Def</u> A fully quantified expression in predicate logic is a <u>theorem</u> iff it is T for every possible meaning of its predicates (akin to a tautology)

$\forall S$

<u>Thm</u> (3.39) Let S be any set. $\forall x \in S : [P(x) \vee \neg P(x)]$

implied $\forall P$

<u>ex</u> $P(x) = $ isEven(x), $S = \mathbb{Z}$.

$\forall x \in \mathbb{Z} : [\text{isEven}(x) \vee \neg \text{isEven}(x)]$

Pf For any $x \in S$, $P(x)$ is defined, and $P(x) = T$ or $P(x) = F$ — defn. of predicate

For any $x \in S$, $P(x) \vee \neg P(x)$ — defn of $\vee, \neg$

$\forall x \in S : [P(x) \vee \neg P(x)]$ — def of $\forall$

Non-Thm (3.40)

$$\left[[\forall x \in S : P(x)] \vee [\forall x \in S : \neg P(x)] \right]$$

note: implied $\forall S$, $\forall P$

disproof

$$[\underline{\forall x \in \mathbb{Z} : \text{isEven}(x)}] \vee [\forall x \in \mathbb{Z} : \neg \text{isEven}(x)]$$

consider $x = 3$ consider $x = 2$

Def Fully quantified expressions φ and ψ are logically equivalent ($\varphi \equiv \psi$, $\varphi \Leftrightarrow \psi$) iff "$\varphi \Leftrightarrow \psi$" is a theorem – that is, they have the same meaning under every interpretation of predicates.

Thm (3.41) $\neg \underline{[\forall x \in S : P(x)]} \Leftrightarrow \underline{[\exists x \in S : \neg P(x)]}$

ex to disprove $\forall x \in \mathbb{Z} : \text{isEven}(x)$, we found $x \in S : \neg \text{isEven}(x)$ ($x = 3$)

This theorem explains why disproof by counterexample works!

Intuition behind proof:

let $S = \{s_1, s_2, s_3, \ldots\}$. Then:

$\neg[\forall x \in S : P(x)]$ given

$\cong \neg[P(s_1) \wedge P(s_2) \wedge P(s_3) \ldots]$ def of $\forall$

$\cong \neg P(s_1) \vee \neg P(s_2) \vee \neg P(s_3) \vee \ldots$ de Morgan's Law

$\cong \exists x \in S : \neg P(x)$ def of $\exists$ $\square$

suppose $S = \emptyset$. $P(x)$ is generic.

$\neg[\forall x \in S : P(x)]$ $[\exists x \in S : \neg P(x)]$

F F

Thm (3.42) $\neg[\exists x \in S : Q(x)] \Leftrightarrow [\forall x \in S : \neg Q(x)]$

Pf let $P(x) = \neg Q(x)$.

$\neg[\forall x \in S : P(x)] \Leftrightarrow [\exists x \in S : \neg P(x)]$ 3.41

$\forall x \in S : P(x) \Leftrightarrow \neg[\exists x \in S : \neg P(x)]$ negating both sides

$\forall x \in S : \neg Q(x) \Leftrightarrow \neg[\exists x \in S : Q(x)]$ subs. $\square$

ex $\neg(\exists x \in \mathbb{R} : x^2 + 1 = 0) \equiv \forall x \in S : x^2 + 1 \neq 0$

Thm (3.43) For $S \neq \emptyset$, $[\forall x \in S : P(x)] \Rightarrow [\exists x \in S : P(x)]$

"if it's true for all, it's true for one"

"if everybody's doing it, then somebody's doing it" ex $\underline{\forall x \in \mathbb{Z} : isEven(2x)} \Rightarrow \underline{\exists x \in \mathbb{Z} : isEven(2x)}$

Pf (direct)

Assume $\forall x \in S : P(x)$. WTS $\exists x \in S : P(x)$.

$\exists a \in S$	since $S \neq \emptyset$
$P(a)$ true	$a \in S$ and assumed $\forall x \in S : P(x)$
$\exists x \in S : P(x)$	it's a! $\square$

Thm $\forall x \in \emptyset : P(x)$ "$P(x)$ is vacuously true"

Pf Aiming for a contradiction, suppose the claim is false.

$\neg [\forall x \in \emptyset : P(x)]$	assumption
$\exists x \in \emptyset : \neg P(x)$	3.41

which is a contradiction, since there are no elements in $\emptyset$. $\square$