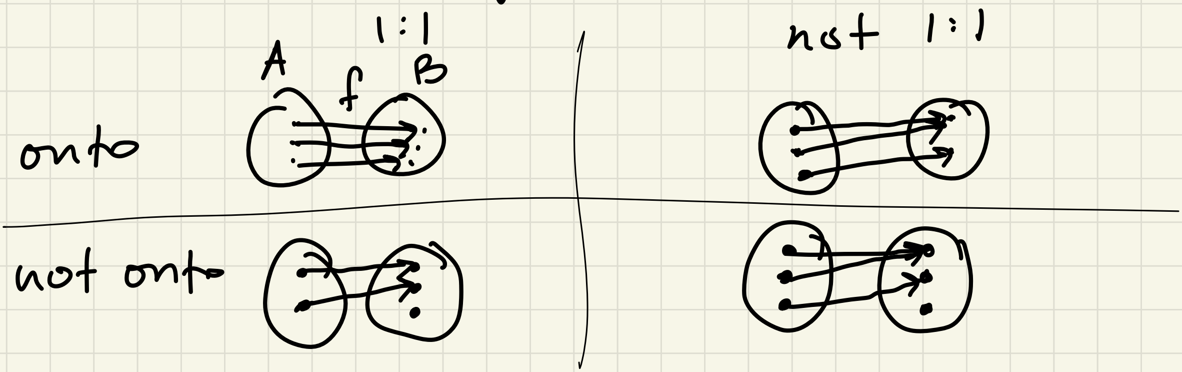
<u>Def</u> A function $f: A \to B$ is

1. onto (surjective) if
$\forall b \in B \ \exists a \in A : f(a) = b$
$\equiv \forall b \in B$, something in A maps to it
$\equiv \forall b \in B$, b shows up in ≥ 1 row of table
$\equiv$ codomain = range

2. one-to-one (injective) if
$1:1$
$\forall a_1, a_2 \in A \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
$\equiv \forall b \in B$, at most 1 thing in A maps to it
$\equiv \forall b \in B$, b shows up in ≤ 1 row of table

3. a bijection if both onto and 1:1
$\forall b \in B$, exactly 1 elt. of A maps to it.



How do we prove that f onto / 1:1?

onto

WTS $\forall b \in B \ \exists a \in A : f(a) = b$.

$\equiv$ if $b \in B$ then $\exists a \in A : f(a) = b$.

Step 1: suppose that $b \in B$.

Step 2: Show that $\exists a \in A : f(a) = b$ by constructing a s.t. $f(a) = b$.

ex recall $s : \mathbb{Z} \to \mathbb{Z}$, $s(x) = x+1$.

Claim: S is onto.

$s(x) = x+1$. have $b \in$. how did I get it? $b-1$.

proof let $b \in \mathbb{Z}$. we need to show that $\exists a \in \mathbb{Z} : s(a) = b$. Consider $a = b-1$. $a \in \mathbb{Z}$ and $s(a) = b-1+1 = b$, as needed. $\square$

This is an example of proof by construction.

not onto:

WTS $\neg(\forall b \in B \ \exists a \in A : f(a) = b)$

$\equiv \exists b \in B \ \forall a \in A \ f(a) \neq b$

construct $b \in B$ s.t. nothing in A maps to it.

ex $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ not onto.

Proof consider $b = -1 \in \mathbb{R}$

$\forall a \in \mathbb{R}: f(a) = a^2$ — def of f

$\forall a \in \mathbb{R}: f(a) \geq 0$ — prop. of ^{2}

$\forall a \in \mathbb{R}: f(a) \neq b$ — $b < 0$

invalid proof that f is onto:

Let $b \in \mathbb{R}$. WTS that $\exists a \in \mathbb{R}: f(a) = b$.
Consider $a = \sqrt{b}$. since $b \in \mathbb{R}$, $\sqrt{b} \in \mathbb{R}$.
Also, $f(a) = (\sqrt{b})^2 = b$.

next time: 1:1

proving $f: A \to B$ 1:1

WTS $\forall a_1, a_2 \in A$ $\quad \underbrace{a_1 \neq a_2}_{p} \Rightarrow \underbrace{f(a_1) \neq f(a_2)}_{q}$

direct proof:

1. assume $a_1, a_2 \in A$ and $a_1 \neq a_2$
2. show that $f(a_1) \neq f(a_2)$

contrapositive:

1. assume $a_1, a_2 \in A$ and $f(a) = f(b)$ ←
2. show $a_1 = a_2$ ←

ex $S : \mathbb{Z} \to \mathbb{Z}$ $\quad S(x) = x+1$

Claim: S is 1:1

Proof Suppose $a_1, a_2 \in \mathbb{Z}$ and $S(a_1) = S(a_2)$.
WTS $a_1 = a_2$.

$a_1 + 1 = a_2 + 1$ — def of S, assumed $S(a_1) = S(a_2)$

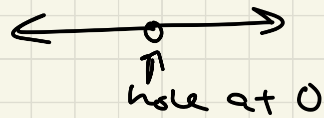
$a_1 = a_2$ — algebra

$\square$

← recall \ = set minus

ex $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ $\quad f(x) = \frac{1}{x} + 1$

domain: real #s except 0



claim: f is 1:1

proof: aiming to prove the contrapositive, assume $a_1, a_2 \in \mathbb{R}\setminus\{0\}$ and $f(a_1) = f(a_2)$.
WTS $a_1 = a_2$.

$\frac{1}{a_1} + 1 = \frac{1}{a_2} + 1$ — def of f, $f(a_1) = f(a_2)$

$\frac{1}{a_1} = \frac{1}{a_2}$ — algebra

$a_1 = a_2$

$\square$

not 1:1

$$\neg\left[\forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)\right]$$

$$\equiv \exists a_1, a_2 \in A : \neg\left[(a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)\right]$$

$$\equiv \exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

$$(\neg(p \Rightarrow q) \equiv p \wedge \neg q)$$

exists a_1, a_2 different but $f(a_1) = f(a_2)$

agrees with our idea of disproof by counter example!

claim:

ex $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ not 1:1

proof: let $a_1 = 2 \in \mathbb{R}$, $a_2 = -2 \in \mathbb{R}$.

$f(a_1) = 2^2 = 4$

$f(a_2) = (-2)^2 = 4$

so $a_1 \neq a_2 \wedge f(a_1) = f(a_2)$

Note: to prove f a bijection, prove both:

f onto

f 1:1

we did this for $s = (x+1)$.