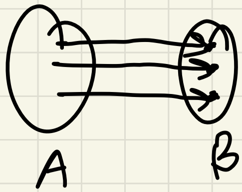
Note:

$f: A \to B$ is a bijection $\Rightarrow |A| = |B|$



$s: \mathbb{Z} \to \mathbb{Z}$ is a bijection

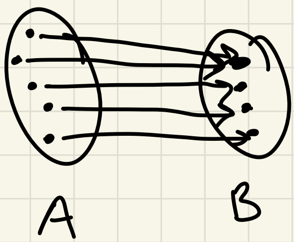
$|\mathbb{Z}| = |\mathbb{Z}|$

$|2\mathbb{Z}| = |\mathbb{Z}|$

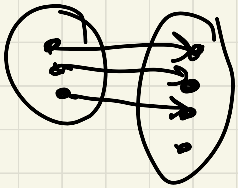
$|\{x \in \mathbb{Z} : 2 \mid x\}|$

$|\mathbb{R}| > |\mathbb{Z}|$ proof by contradiction
bijection

$f: A \to B$ is onto $\Rightarrow |A| \geq |B|$



$f: A \to B$ is 1:1 $\Rightarrow$ $|A| \leq |B|$



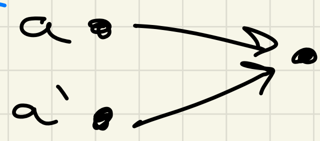
contrapos.

$|A| > |B| \Rightarrow f$ is not 1:1

Theorem 9.13 (Pigeonhole Principle) PHP

Let A, B be sets and $f: A \to B$ a function.

If $|A| > |B|$, then there are 2 distinct $a, a' \in A$ s.t. $f(a) = f(a')$.



$$|A| > |B| \Rightarrow \exists\, a, a' \in A : (a \neq a') \wedge f(a) = f(a')$$

<u>Proof</u> The PHP is contrapos. of

The "pigeon" way of thinking about this:

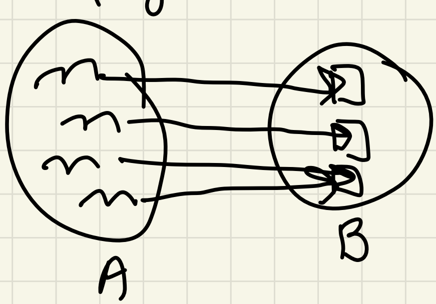
Suppose $A = n+1$ pigeons

$B = n$ cubbies

each pigeon flies to a cubby.

if $f(\text{pigeon } a)$ = pigeon a's cubby

≥ 2 pigeons share a cubby



<u>claim</u> among 13 people, ≥ 2 share a birth month.

<u>proof</u> let A (pigeons) be the set of 13 people.

let B (pigeon holes) be the set of 12 months.

$f: A \to B$ defined by $f(\text{person } a) =$ a's birth month.

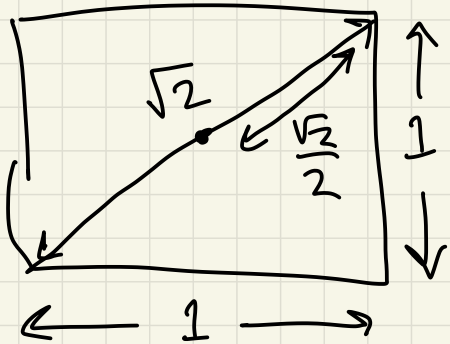
is f a function?

1) each a has birth month
2) each person a has only one birth month
3) every birth month is in the set of months

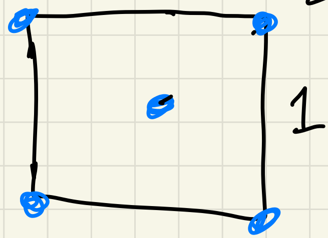
note that $\left.\begin{array}{l} |A| = 13 \\ |B| = 12 \end{array}\right\} \Rightarrow |A| > |B|$

Thus by PHP $\exists a_1, a_2 \in A$ s.t. $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

So there are 2 distinct people a_1 and a_2 s.t. a_1's birth month and a_2's birth month are same.



draw 5 points greater than $\frac{\sqrt{2}}{2}$ apart



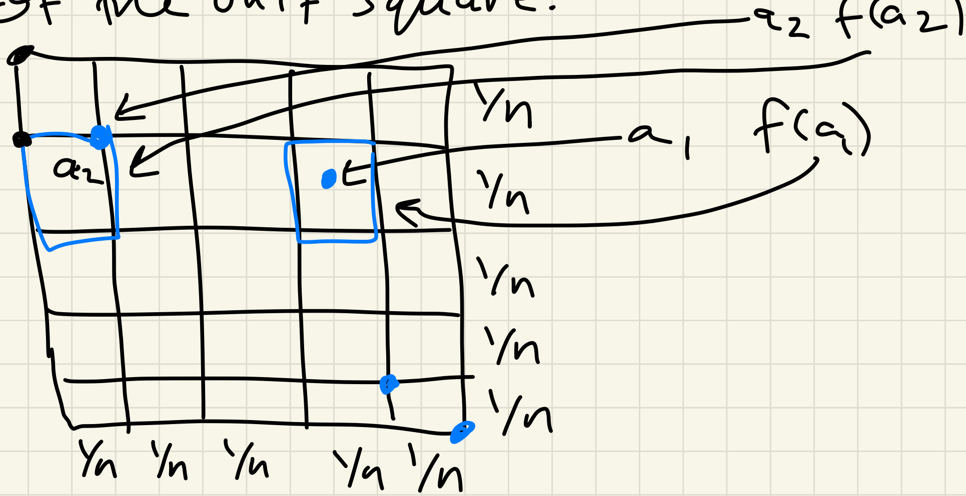
<u>claim</u> (9.36) Let $n \geq 0$, integer.

Suppose $\exists\, n^2+1$ points in unit square.
then $\exists$ 2 points within $\sqrt{2}/n$ of each other.

<u>ex</u> Let $n=2$. $n^2+1=5$.

<u>Pf</u> Let A be the set of n^2+1 points.

Let B be the set of n^2 $1/n \times 1/n$ boxes of the unit square.



let $f: A \to B$ $\quad f(\text{point } a) =$ the $1/n \times 1/n$ box that contains a.

on shared boundaries, map to below + left box.

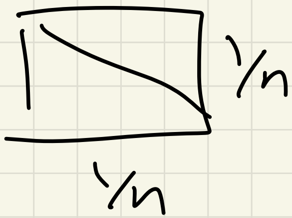
is f a function? Check 1, 2, 3.

1: is f defined for all $a \in A$?

Note: $\left.\begin{array}{l} |A| = n^2+1 \\ |B| = n^2 \end{array}\right\} \Rightarrow |A| > |B|$

By PHP, $\exists a_1, a_2 \in A$ s.t. $a_1 \neq a_2$, $f(a_1) = f(a_2)$
That is, $\exists$ 2 distinct points a_1, a_2 s.t.
a_1, a_2 are within some $1/n \times 1/n$ subsquare.

within a $1/n \times 1/n$ subsquare, the farthest that 2 points can be is $\frac{\sqrt{2}}{n}$:



$$\sqrt{\left(\tfrac{1}{n}\right)^2 + \left(\tfrac{1}{n}\right)^2}$$

$$= \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n}.$$