<u>Recursion</u> is a common strategy for solving CS problems.

- take a problem instance
- split it into subproblems
- until they are small

<u>ex</u> binary search

Problem: find an element in a sorted array

$$A = \langle \underbrace{a_1, a_2, a_3,}\ \underbrace{\dots\ \ , a_n}\rangle$$

$$\begin{tikzcd}
 & & & \\
 & A_l \arrow[dl] \arrow[dr] & & A_r \arrow[dl] \arrow[dr] & \\
A_{l_l} & & A_{l_r} \quad A_{r_l} & & A_{r_r} \\
 & \vdots & & \vdots &
\end{tikzcd}$$

base case: only one element.

<u>Mathematical induction</u> is a proof technique that is analogous to recursion.

<u>ex</u> to prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$,

we prove that the formula holds for $n=0$ (base case) and that if it holds for some $n \geq 1$, then it holds for $n+1$.

<u>Def</u> Let P be a predicate concerning ints ≥ 0. To give a proof by mathematical induction that $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove 2 things:

(1) Base case: prove $P(0)$.

(2) Inductive case: $\forall n \geq 1$, prove

$$P(n-1) \Rightarrow P(n)$$

If we do (1) and (2), we've proved $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$. why?

<u>ex</u> suppose we have proven $P(0)$ and $P(n-1) \Rightarrow P(n)$. These establish $P(3)$

<u>Proof</u> WTS $P(3)$.

<u>statement</u>	reasoning
$P(0)$	we proved it (base case)

$P(0) \Rightarrow P(1)$

$P(1)$

$P(1) \Rightarrow P(2)$

$P(2)$

$P(2) \Rightarrow P(3)$

$P(3)$

$n=1$ for $P(n-1) \Rightarrow P(n)$

bc $P(0)$ (modus ponens)

bc $P(n-1) \Rightarrow P(n)$

modus ponens

bc $P(n-1) \Rightarrow P(n)$

(1) Base case: prove $P(0)$.

(2) Inductive case: $\forall n \geq 0$, prove

$P(n) \Rightarrow P(n+1)$

claim, $\underset{n \in \mathbb{Z}}{\forall n \geq 0}$, $\overbrace{\sum_{i \in \{0,1,\ldots,n\}} 2^i}^{\text{LHS}} = \overbrace{2^{n+1}-1}^{\text{RHS}}$

$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

ex

n	LHS	RHS
0	$2^0 = 1$	$2^1 - 1 = 2 - 1 = 1$
1	$2^0 + 2^1 = 1 + 2 = 3$	$2^2 - 1 = 4 - 1 = 3$
2	$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$	$2^3 - 1 = 8 - 1 = 7$

Steps to prove a "$\forall n \geq 0$, ______" statement using mathematical induction:

① Clearly state $P(n)$ and that your proof is by induction and which variable you are performing induction over.

② Prove $P(0)$ (base case)

③ Prove $\forall n \geq 1$, $P(n-1) \Rightarrow P(n)$ (inductive case)

$P(n-1)$: inductive hypothesis.

claim: $\forall n \geq 0, \ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Proof

① We define $P(n)$ to mean that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

$$\underset{n \geq 0}{P(n)} = \begin{cases} T & \text{if } \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \\ F & \text{otherwise} \end{cases}$$

We show by induction that $\forall n \geq 0 : P(n)$.
We use induction over n.

② For the base case, we WTS $P(0)$.
i.e., WTS $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

Let's check: $2^0 = 1$ and $2^1 - 1 = 1$,
so $P(0)$ holds.

(3) For the inductive case, we need to prove $\forall n \geq 1 : P(n-1) \Rightarrow P(n)$.

Assume $P(n-1)$. (inductive hypothesis)

$$\sum_{i=0}^{n-1} 2^i = 2^{(n-1)+1} - 1 \quad (*).$$

WTS $P(n)$. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

LHS = RHS

$$\begin{aligned} LHS = \sum_{i=0}^{n} 2^i &= \left[\sum_{i=0}^{n-1} 2^i\right] + 2^n && \text{def. of summations} \\ &= \left[2^{(n-1)+1} - 1\right] + 2^n && \text{subs. w/ } (*) \text{ (applying IH)} \\ &= 2^n - 1 + 2^n && \\ &= 2 \cdot 2^n - 1 && \text{algebra} \\ &= 2^{n+1} - 1 && \\ &= RHS && \end{aligned}$$

So we have shown $P(n)$.

We've shown $P(0)$ and $P(n-1) \Rightarrow P(n)$,
so by the principle of mathematical

induction, $P(n)$ holds $\forall n \geq 0$.

Bogus proof:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

$\vdots$

<u>claim</u>: $\forall n \geq 0$, $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

$$0+1+2+\cdots n = \frac{n(n+1)}{2}$$

<u>ex</u>	<u>n</u>	$0+1+2+\cdots+n$	$\frac{n(n+1)}{2}$
	1	$0+1=1$	$\frac{1(2)}{2}=1$
	2	$0+1+2=3$	$\frac{2(3)}{2}=3$
	5	$0+1+2+3+4+5=15$	$\frac{5(6)}{2}=\frac{30}{2}=15$

<u>Proof</u> We prove using mathematical induction.

① We define the predicate $P(n)$ to be

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

We prove that $\forall n \geq 0 : P(n)$ using mathematical induction over n.

② For the base case, consider $n=0$. Then $\sum_{i=0}^{n} i = 0$ and $\frac{n(n+1)}{2} = 0$, so $P(0)$ holds.

③ For the inductive case, we prove that $\forall n \geq 1$, $P(n-1) \Rightarrow P(n)$.

Assume $P(n-1)$. That is, $\sum_{i=0}^{n-1} i = \frac{(n-1)(n-1+1)}{2}$

WTS $P(n)$. That is, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

$$\sum_{i=0}^{n} i = \sum_{i=0}^{n-1} i + n \qquad \text{algebra / def. of sum}$$

$$= \frac{(n-1)(n-1+1)}{2} + n \qquad \text{subs. w/ IH}$$

$$= \frac{(n-1)(n)}{2} + n$$

$$= \frac{n(n-1)}{2} + \frac{2n}{2} \qquad \text{algebra}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n(n-1+2)}{2}$$

$$= \frac{n(n+1)}{2}$$

write names of group of 3 or more + turn for bonus.

There is a typo in (3).
should be $P(n-1) \Rightarrow P(n)$

not $n-1$

HINTS once you get to (3),
inductive step:

- start w/ LHS (2^n).
- try to manipulate so you get something from the inductive hypothesis.

note that $n \geq 4 \Rightarrow n^2 \geq 4n$,
so $\underbrace{n^2 - 4n}_{} \geq 0$