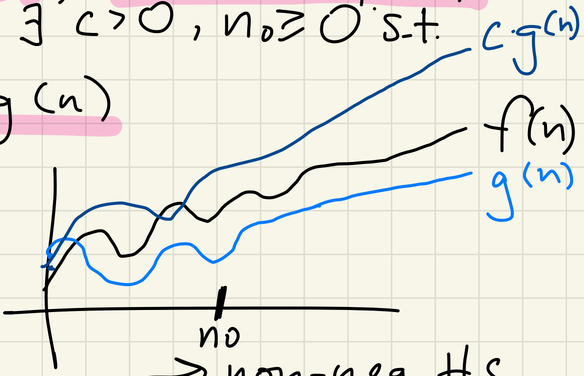
Recall

"has property that"

For functions $f(n)$, $g(n)$ $f(n) \overset{\downarrow}{=} O(g(n))$

"f is big O of g"   if $\exists\ c > 0,\ n_0 \geq 0$ s.t.

$\forall n \geq n_0 : f(n) \leq c \cdot g(n)$



$\rightarrow$ non-neg #s

ex	n_0	c
$\underset{f(n)}{\underset{\uparrow}{n}} = O(\underset{g(n)}{\underset{\uparrow}{n}})$	0	3
$2n = O(n)$	0	3
$n+8 = O(n)$	4	3
$10 = O(n)$	1	10

Logarithms for positive real # $b \neq 1$ and real # $x > 0$, $\log_b x$ is the real number y s.t. $b^y = x$.

$\log_4 16$ means "the number we need to raise 4 to to get 16"

$= 2$

$\log_{10} 100$

$= 2$

$\log_2 8 = 3$ $2^3 = 2 \cdot 2 \cdot 2 = 8$

Lemma 6.7 Let $b > 1$ and $k \geq 0$.

$$\log_b(n^k) = O(\log n)$$

↑ missing base... because it doesn't matter

$f(n) = O(\log n)$

ex $\log_{10} n = O(\log_2(n))$ by Lemma 6.7.

Lemma 6.7, intuitively: base, exponents in logs don't matter asymptotically.

Proof WTS $\exists\, c > 0, n_0 \geq 0$ s.t. $\forall n \geq n_0$:

$$\log_b(n^k) \leq c \log_a n$$

↗ can be anything, so we'll drop later

Note that

$$\log_b(n^k) = k \log_b(n)$$

log rule: exponent

$$= k \frac{\log_a(n)}{\log_a(b)}$$

log rule: change of base

Now take $n_0 = 1$, $c = \dfrac{k}{\log_a b}$.

$$\forall n \geq 1,\ \overset{f(n)}{\downarrow}\log_b(n^k) = \underbrace{\frac{k \log_a(n)}{\log_a(b)}}_{} \leq \underbrace{\overset{c}{\downarrow}\frac{k}{\log_a(b)} \cdot \overset{g(n)}{\downarrow}\log_a(n)}_{}$$

so $\log_b(n^k) = O(\log_a n)$.

Since a could have been anything, we drop it.

<u>Q</u> $3^n = O(2^n)$? $k^n = O(\ell^n)$? $\forall k, \ell$

<u>Lemma</u> Let $b, d > 1$. If $d < b$ then $b^n \neq O(d^n)$.

<u>ex</u>
Let $d = 2$, $b = 3$. $d < b$ because $2 < 3$.

$3^n \neq O(2^n)$.

"3^n is asymptotically larger than 2^n"

This lemma tells us that the base of an exponent does matter, asymptotically.

$n^3 \neq O(n^2)$

Lemma 6.2 Asymptotic Equivalence of Max + Sum

$f(n) = O(g(n)+h(n)) \iff f(n) = O(\max(g(n),h(n)))$

ex $f(n) = n^2+n = O(n^2+n)$

By Lemma 6.2, $f(n) = O(\max(n^2, n))$

$f(n) = O(n^2)$

$g(n) = n^2$, $h(n) = n$

This lemma is what allows us to drop lower-order terms.

Proof Because Lemma 6.2 is an $\iff$, we prove each direction separately.

$(\Rightarrow)$ $f(n) = O(g(n)+h(n)) \Rightarrow f(n) = O(\max(g(n),h(n)))$

Assume $f(n) = O(g(n)+h(n))$. WTS $f(n) = O(\max(g(n),h(n)))$.

$\exists\, c>0,\ n_0 \geq 0 : \forall n \geq n_0 :$

$$
\begin{aligned}
f(n) &\leq c \cdot [g(n)+h(n)] \\
&\leq c \cdot [\max(g(n),h(n)) + h(n)] \\
&\leq c \cdot [\max(g(n),h(n) + \max(g(n),h(n)] \\
&= c \cdot [2 \cdot \max(g(n),h(n))]
\end{aligned}
$$

$\rightarrow = 2c \cdot \max(g(n), h(n))$

choose $c' = 2c$, $n_0' = n_0$.

goal: find some $c' > 0$, $n_0' \geq 0$ s.t.

$$\forall n \geq n_0' : f(n) \leq c' \cdot [\max(g(n), h(n)]$$

($\Leftarrow$) in book

some input goes here ↓

Lemma 6.3 Transitivity of $O(\cdot)$

If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

ex

$f(n) = 3n$
$g(n) = 2n+2$
$h(n) = 4n^2$

$f(n) = O(g(n))$
$g(n) = O(h(n))$ $\Rightarrow$ $f(n) = O(h(n))$

$$3n = O(4n^2)$$

Lemma 6.4 Addition and multiplication preserve $O(\cdot)$-ness.

If $f(n) = O(h_1(n))$ and $g(n) = O(h_2(n))$, then

$$f(n) + g(n) = O(h_1(n) + h_2(n))$$
$$f(n) \cdot g(n) = O(h_1(n) \cdot h_2(n))$$

Not true for $-$, $/$

ex $f(n) = n^3$

$\uparrow$ against Lemma 6.4 for −

$$\left.\begin{array}{l} g(n) = n^2 \\ f(n) = O(n^3) \\ g(n) = O(n^3) \end{array}\right]$$

$$f(n) - g(n) = \underline{n^3 - n^2}$$

$$n^3 - n^3 = 0$$

is $n^3 - n^2 = O(\underset{\substack{\uparrow \\ \text{zero}}}{0})$?

ex of Lemma 6.4 in action

$$n^3 + n^2 = O(n^3 + n^3)$$

Lemma 6.5 let $p(n) = \sum_{i=0}^{k} a_i n^i$

$$= a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$$

be a polynomial. Then $p(n) = O(n^k)$.

ex $p(n) = 3n^3 - 2n + 5 = O(n^3)$

Common Distinct Functions

name	$f(n)$	$O(\cdot)$
constant	$c \in \mathbb{R}^{\geq 0}$	$O(1)$
log	$\log_b n, b \in \mathbb{R}^{>1}$	$O(\log n)$
linear	$c \cdot n, c \in \mathbb{R}^{>0}$	$O(n)$
n-log-n	$n \log_b n$	$O(n \log n)$

quadratic	deg-2 polynomial $c_2n^2 + c_1n + c_0$	$O(n^2)$
cubic	deg-3 poly.	$O(n^3)$
$\vdots$		
deg k poly.		$O(n^k)$
exponential	2^n	$O(2^n)$
	3^n	$O(3^n)$
	4^n	$\vdots$

let's think about $n!$