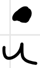
(pronounced "kleek")

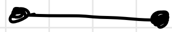
**Def** A <u>complete graph</u> or <u>clique</u> is an undirected graph $G = (V, E)$ s.t.

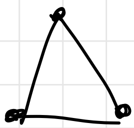
$$\forall u, v \in V \quad u \neq v \Rightarrow \{u, v\} \in E$$

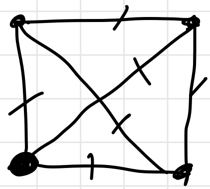
The clique on $n$ nodes is denoted $K_n$.

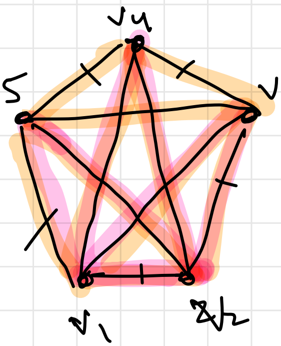
**ex** $K_1$ 

0

$K_2$  1

$K_3$  3

$K_4$  6

$K_5$ 

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**Q** What is the relationship btwen $n = |V|$ and $m = |E|$ for $K_n$?

**A** # of ways to choose 2 nodes out of $n$

sum of natural #s less than $n$

$$\frac{n(n-1)}{2}$$

Claim $K_n$ has $\frac{n(n-1)}{2}$ edges.

Proof #1 We give a way to count the edges and show that it gives $\frac{n(n-1)}{2}$.

label the nodes $v_1, v_2, \dots, v_n$. Starting w/ $v_1$, count the uncounted edges and add to the total.

$v_1$ has $n-1$ uncounted edges
$v_2$ has $n-2$ uncounted edges
$\vdots$
$v_{n-1}$ has $1$ uncounted edge
$v_n$ has $0$ uncounted edges

$$|E| = (n-1) + (n-2) + \dots 1 + 0 = \frac{n(n-1)}{2}$$

Proof #2 In $K_n$ every node has deg. $n-1$. So,

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (n-1) = \underline{n(n-1)}$$

But by the handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$n(n-1) = 2|E|$$

$$\frac{n(n-1)}{2} = |E| = m$$

**Proof #3** Let $P(n)$ denote that $K_n$ has $\frac{n(n-1)}{2}$ edges. We prove $\forall n \geq 1 : P(n)$ using induction over $n$.

Base Case: $n = 1$.

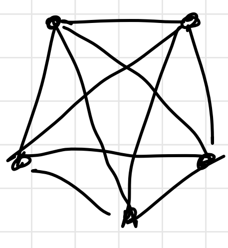
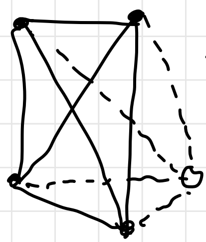
$K_1$ • has 0 edges.

$\frac{1(1-1)}{2} = 0$ so $P(1)$ holds.

Inductive case: we WTS $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$

Assume $P(n-1)$. That is, assume $K_{n-1}$ has $\frac{(n-1)(n-2)}{2}$ edges.

Now, consider an arbitrary clique $K_n$. Let $K_n'$ be the graph created by removing one node and all its incident edges from $K_n$.

(example: $K_5$ :  $K_5'$  )

Note that $K_n' = K_{n-1}$

Goal: # edges $K_n = \frac{n(n-1)}{2}$

# edges of $K_n$ = # of edges of $K_{n-1}$ + # of edges have to add to $K_{n-1}$ to get $K_n$

IH ↘

$$= \frac{(n-1)(n-2)}{2} \qquad (n-1)$$

$$= \frac{n^2-3n+2}{2} + \frac{2(n-1)}{2}$$

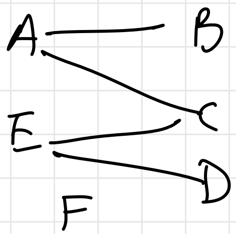
$$= \frac{n^2-3n+2+2n-2}{2}$$

$$= \frac{n^2-n}{2}$$

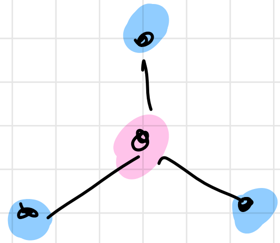
$$= \frac{n(n-1)}{2}$$

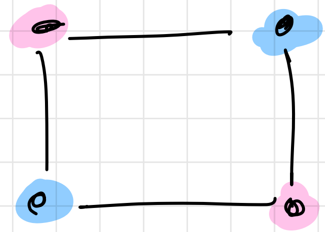
__Def__ A __bipartite graph__ $G=(L\cup R, E)$ s.t. $L\cap R=\emptyset$ ($L$, $R$ disjoint) and $E \subseteq \{\{l,r\} : l\in L, r\in R\}$

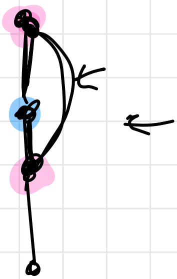
__ex__

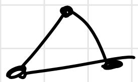



$L=\{A, E\}$
$R=\{B, C, D\}$







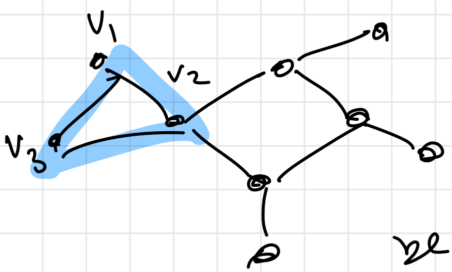


<u>Claim</u> If $G$ contains a $\Delta$ ($K_3$), then it is not bipartite.

<u>Proof</u> Aiming for a contradiction, suppose that $G$ contains a $\Delta$ and it is bipartite.

Let $v_1, v_2, v_3$ be the nodes of the $\Delta$.

(example:

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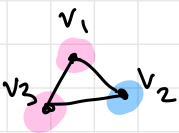


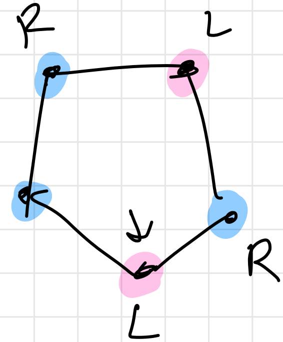

because we would reorder $v_1, v_2, v_3$

Without loss of generality, suppose $v_1 \in L$ and $v_2 \in R$. Since $v_2 \in R$, $v_3 \in L$. But there is an edge from $v_1$ to $v_3$ and both are in $R$, which contradicts that $G$ is bipartite.

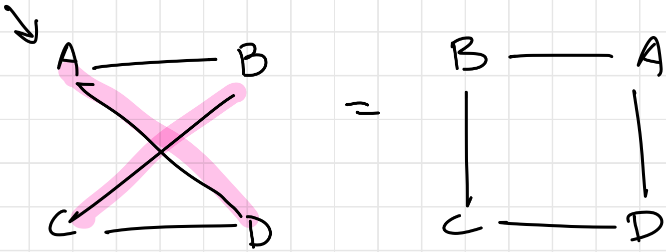
Q other dir?

(claim) If a graph is not bipartite, then it has a $\Delta$.

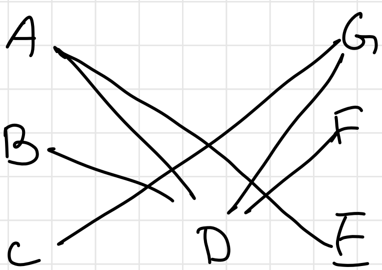
Disproof by counterexample:

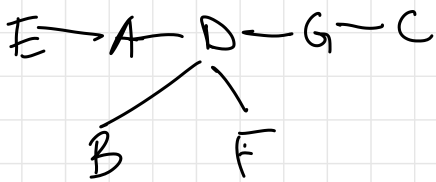



**Def** A graph is planar if we can draw it in the plane w/out edge crossings.



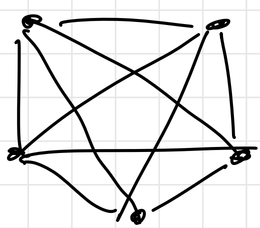

note: 2 graph are equal if verts same, edges same

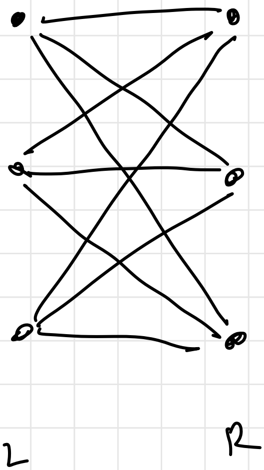



$K_4$ is planar

$K_5$ is not






complete bipartite graph on 3 nodes

$K_{3,3}$