Randomness + Probability in CS:

- average-case runtime analysis
- randomized algorithm
- data structures using randomness
- modeling real-world phenomena
- security

But first, let's start learning to count!

Sum rule: if $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

Product rule: the number of pairs $\langle x, y \rangle$ with $x \in A$, $y \in B$ is $|A| \cdot |B|$.

$$|A \times B| = |A| \cdot |B|$$

ex A restaurant has 2 lunch specials.

1. soup or a salad
2. soup and a salad

$A = \{\text{chicken noodle}, \dots\}$

$B = \{\text{caesar}, \dots\}$

How many options for 1? $|A| + |B|$

2? $|A| \cdot |B|$

More general product rule:

$A_1, A_2, \dots, A_k$ sets

$$|A_1 \times A_2 \times A_3 \cdots A_k| = |A_1| \cdot |A_2| \cdots |A_k|$$

ex How many 32-bit strings are there?

$$\underbrace{0100\cdots 11}_{\text{32-bit string}} \qquad 2^{32}$$

$$|\{0,1\} \times \{0,1\} \times \{0,1\} \cdots \{0,1\}|$$
$$= |\{0,1\}^{32}| = |\{0,1\}|^{32} = 2^{32}$$

Def Given some random process, the sample space S is the set of all possible outcomes.

A probability function $Pr : S \to \mathbb{R}$ describes the fraction of the time that $s \in S$ occurs.

we use square brackets for prob. functions

$$\sum_{s \in S} Pr[s] = 1$$

$$Pr[s] \geq 0 \quad \forall s \in S$$

ex flipping a coin $\quad S = \{\text{heads}, \text{tails}\}$

$Pr[\text{heads}] = 0.5$ $\quad$ sum to 1
$Pr[\text{tails}] = 0.5$ $\quad$ $Pr[s] \geq 0$

ex drawing a card from a deck

$$S = \{2 \text{ clubs}, 3 \text{ clubs}, \cdots\}$$

$Pr[s] = \frac{1}{52} \quad \forall s \in S$

ex flipping 2 coins

$S = \{ \langle H, H \rangle, \langle H, T \rangle, \langle T, H \rangle, \langle T, T \rangle \}$

each has probability 0.25

$Pr[s] = 0.25 \quad \forall s \in S$

all of these had uniform probability functions

ex Let $S = \{0, 1, 2, \cdots, 7\}$. Choose from S by flipping 7 coins & counting # of H.

HHHHHHH $\rightarrow$ 7 $\quad Pr[7] = 0.0078$

$Pr[4] = 0.2734$

Def A set of outcomes is an event.

$E \subseteq S, \quad Pr[E] = \sum_{s \in E} Pr[s]$

ex when flipping 2 coins, the probability that at least one is H is

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3/4$

ex when drawing 1 card, the prob. that it is an ace

$1/52 + 1/52 + 1/52 + 1/52 = 1/13$

Theorem 10.4 Properties of event probabilities

Let S be a sample space and $A \subseteq S$, $B \subseteq S$ events. Let $\bar{A} = S - A$ be the complement of A.

$$Pr[\bar{A}] = 1 - Pr[A]$$
$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

ex When drawing one card, what is the probability that it's not an ace?

$S = \{\text{all cards}\}$
$A = \{\text{A clubs, A spades, A hearts, A diamonds}\}$

$$Pr[\bar{A}] = 1 - Pr[A] = 1 - 4/52 = 48/52$$

not an ace

ex When drawing 1 card what's the prob. that it's a Q or a heart?

$A = \{\text{queens}\}$

$B = \{\text{all hearts}\}$

$A \cap B = \{\text{Q of hearts}\}$

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$
$$= 4/52 + 13/52 - 1/52 = 16/52$$