So far:

whether something occurs

Now: how many?

<u>ex</u> how many times do we have to flip a coin to get 100 heads?

In a randomly sorted array, ~~for~~ how many slots is $A[i] < A[i+1]$?

<u>def</u> A <u>random variable</u> X assigns a numerical value to every outcome of a sample space S.

$X : S \to \mathbb{R}$

<u>ex</u> Suppose we flip a coin 3 times

$S = \{H, T\}^3 = \{\langle H, H, H \rangle, \langle H, T, H \rangle, \cdots\}$

$Pr[s] = \frac{1}{8} \quad \forall s \in S$

Let X = # heads
Y = # of consecutive T (from start)

$X(TTT) = 0$ $\quad$ $X(T\cancel{H}H) = 2$

$Y(TTT) = 3$ $\quad$ $Y(THH) = 1$

ex Let S be the set of all English words

Let L = # letters of a word

$L(\text{computer}) = 8$

Def The expectation of a random variable X denoted $E[X]$, is the average value of X.

$$E[X] = \sum_{s \in S} X(s) \cdot \Pr[s]$$

$$= \sum_{\substack{y:\ \exists s \in S: \\ X(s) = y}} y \cdot \Pr[X = y]$$

ex Counting heads in 3 coin flips

X = # heads

expected # heads $= E[X] = \sum_{x \in S} X(x) \cdot \Pr[x]$

$$= X(HHH) \cdot \Pr[HHH] + X(HHT)\Pr[HHT] \cdots$$

$$= \frac{1}{8}\Big(X(\underline{HHH}) + X(\underline{HH}T) + X(\underline{H\ T\ H}) + X(\underline{H\ T\ T}) + X(THH) + X(THT) + X(TTH) + X(\underline{TTT})\Big)$$

$$= \frac{1}{8}(3 + 2 + 2 + 1 + 2 + 1 + 1 + 0)$$

$$= \frac{1}{8}(12) = 1.5$$

$$E[X] = \sum_{y \in \{0,1,2,3\}} y \cdot Pr[X(x) = y]$$

$$= 0 \cdot Pr[0 \text{ heads}] + 1 \cdot Pr[1 \text{ head}] + 2 \cdot Pr[2 \text{ heads}] + 3 \cdot Pr[3 \text{ heads}]$$

$$= 0 \cdot \frac{1}{8} + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{12}{8}$$

<u>Thm</u> Linearity of Expectation

Let S be a sample space and $X : S \to \mathbb{R}$, $Y : S \to \mathbb{R}$ be any two random variables.

Then $E[X+Y] = E[X] + E[Y]$