Def A <u>proposition</u> (statement, claim) is a statement that is either always true or always false. For a prop., its <u>truth value</u> is its truth or falsity.

True?

ex y $p =$ $2+2=4$. T

prop? x $q =$ 33 is a prime number. F

x $r =$ Waded Cruzado is currently the president of MSU. T

y Every integer greater than 2 can be written as the sum of two primes. ??

no $1+2+3+4=x$

no Don't forget to do the week 1 drills!

p is true and q is false.

In this class, our task is to learn and practice methods of proving propositions T or F.

Def A <u>proof</u> is a convincing argument that a proposition is true. A <u>disproof</u> is an argument that a prop. is false

What a proof looks like depends on the author and the audience.

Claim (from book example 4.10)

Any positive integer n is divisible by 4 if and only if its last two digits are divisible by 4.

Is it a proposition? Yes!

Step 1: make sure you understand the claim.

- positive integer — 1, 2, 3, ... not 0, −5, 2.5
- divisible by 4 — $n = 4k$ for int k. $12 = 4 \cdot 3$, but 9 ... ?
- last two digits — 1234, 04, → tens and ones
- if and only if — Part 1 implies part 2 and part 2 implies part 1

If a positive integer n is divis. by 4 then its last two digits are div. by 4.

If the last two digits of integer n are div. by 4 then n is div. by 4.

Step 2: do some examples.

n	last 2 digits	n div by 4?	last 2 div by 4?
20	20	$20 = 4 \cdot 5$ Y	Y
17	17	N	N
100	0	$25 \cdot 4$ Y	$0 \cdot 4$ Y
131	31	N	N

Step 3: are there any special cases you can already prove?

Say n is a multiple of 100. Its last two digits are always 00, so div. by 4. And all multiples of 100 are div. by 4. Nice!

<u>Proof</u> Let $d_k, d_{k-1}, \dots, d_1, d_0$ denote the digits of n.

$\Rightarrow$ "this implies that"

$$\Rightarrow \quad n = d_0 + 10d_1 + \cdots + 10^{k-1}d_{k-1} + 10^k d_k$$

by def. of base 10.

$$\Rightarrow \quad n = d_0 + 10d_1 + 100\left(d_2 + 10d_3 + \cdots + 10^{k-3}d_{k-1} + 10^{k-2}d_k\right)$$

by factoring out 100

$$\Rightarrow \quad n = d_0 + 10d_1 + 25 \cdot 4\left(d_2 + 10d_3 + \cdots + 10^{k-3}d_{k-1} + 10^{k-2}d_k\right)$$

because $100 = 25 \cdot 4$

$$\Rightarrow \quad \frac{n}{4} = \frac{(d_0 + 10d_1)}{4} + 25\left(d_2 + 10d_3 + \cdots + 10^{k-3}d_{k-1} + 10^{k-2}d_k\right)$$

by dividing both sides by 4

$\Rightarrow \frac{n}{4}$ is an int if and only if

$$\frac{(d_0 + 10d_1)}{4} + 25\left(d_2 + 10d_3 + \cdots + 10^{k-3}d_{k-1} + 10^{k-2}d_k\right) \text{ is int.}$$

because both equal

$\Rightarrow \frac{n}{4}$ is int. iff (if and only if) $\frac{d_0 + 10d_1}{4}$ is int., because the products, sums, and differences of ints are also ints.

$\Rightarrow \frac{n}{4}$ is int iff $d_0 + 10d_1$ is div. by 4

$\Rightarrow$ n is div. by 4 iff its last two digits are div. by 4.

<u>Def</u> A <u>direct proof</u> starts from known facts or definitions and repeatedly applies logical deduction to derive new facts and end up with the claim.

from book example 4.11

<u>claim</u> If x and y are rational, then $x \cdot y$ is rational.

terms : rational $x = \frac{n}{d}$ where n, d ints $d \neq 0$

examples:

x	y	xy	x, y rational	xy rat.
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	T	T
π	2	2π	F	F

In this proof, we will see the two-column format for writing a proof.

<u>proof</u> Start by assuming that x and y are rational. WTS (want to show) that xy is rational.

statement	reasoning
$x = \frac{n_x}{d_x}$, $y = \frac{n_y}{d_y}$ where n_x, d_x, n_y, d_y are integers and $d_x \neq 0$ and $d_y \neq 0$	by def. of rational
$xy = \frac{n_x n_y}{d_x d_y}$	by substitution
$xy = \frac{n}{d_x d_y}$ where n is an int.	because product of ints is int.
$xy = \frac{n}{d}$ where d is a nonzero int.	because product of nonzero ints is a nonzero int
xy is rational	by def. of rational

$\boxtimes$

Q: is the converse true?

if p then q ← prop.

if q then p ← converse of prop.

our original prop:

if x, y rational then xy rational.

so the converse is:

if xy rational, then x, y rational.

Let's <u>disprove by a counterexample</u>

Give an x, y so that xy rational but x, y are not.

both irrational

$x = \pi$, $y = 1/\pi$. $xy = 1$. ← rational

$x = \pi$, $y = 0$. $xy = 0$. ← rational

irrational

<u>Def</u> A <u>disproof by counterexample</u> constructs an example for which the claim is false and explains why it is false.