Warmup: consider the relation M_5, $= R$
"equivalent mod 5" $\mathbb{Z}$

$\swarrow$

%5, remainder when dividing by 5

11 mod 5 = 1
10 mod 5 = 0

$M_5 = \{\langle n, m\rangle \in \mathbb{Z}\times\mathbb{Z} : n \bmod 5 = m \bmod 5\}$

11 mod 5 = 26 mod 5 $\quad \langle 11, 26\rangle \in M_5$

$11\, M_5\, 26$

$a R b$

- reflexive: $\forall a \in A : a R a$ $\quad$ a

yes.

ex: 11 mod 5 = 11 mod 5 $\quad 11\, M_5\, 11$

proof: WTS $\forall a \in \mathbb{Z}$: $a M_5 a$.

a mod 5 = a mod 5 so $a M_5 a$. $\square$

- irreflexive: $\forall a \in A : a \not R a$

no. ~~dis~~proof by counterexample: 5 mod 5 = 5 mod 5, so $5 M_5 5$.

- symmetric: $\forall a, b \in A : a R b \Rightarrow b R a$. $\quad$ a $\quad$ b

Suppose $a, b \in \mathbb{Z}$. Assume $a M_5 b$. WTS $b M_5 a$.

$a M_5 b$	by assumption
a mod 5 b = b mod 5 a	def. of M_5
b mod 5 a = a mod 5 b	def. of =
$b M_5 a$	def. of M_5

$\square$

- anti-symmetric:
  $\forall a,b \in A: (aRb \wedge bRa) \Rightarrow (a=b)$ $\quad \cancel{a \rightleftarrows b}$

$11 \, M_5 \, 26$, $26 \, M_5 \, 11$ $\quad 26 \neq 11$.
(disproof by counterexample.) $\quad a \to b \to c$

- transitive: $\forall a,b,c \in A: aRb \wedge bRc \Rightarrow aRc$

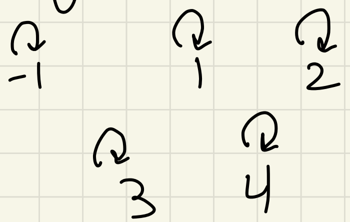
Let $a,b,c \in \mathbb{Z}$. Suppose $a \, M_5 \, b$ and $b \, M_5 \, c$.
WTS $a \, M_5 \, c$.

$a \bmod 5 = b \bmod 5 \wedge b \bmod 5 = c \bmod 5$	def. of M_5
$a \bmod 5 = c \bmod 5$	subs
$a \, M_5 \, c$	def. of M_5 $\square$

Def A binary relation R is an <u>equivalence relation</u> if it is reflexive, symmetric, and transitive.

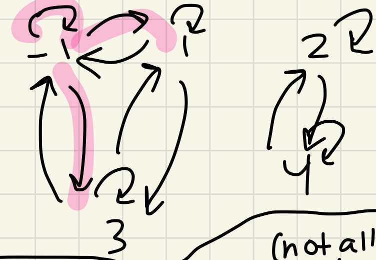
<u>ex</u> M_5 is an equivalence relation.

Consider $A = \{-1, 1, 2, 3, 4\}$.

Relation	graph	equivalence classes
$\langle -1,-1 \rangle$ $\langle 1,1 \rangle$ $\langle 2,2 \rangle$ $\langle 3,3 \rangle$ $\langle 4,4 \rangle$ $=$		$[-1] = \{-1\}$ $\vdots$ $[4] = \{4\}$

same parity

$\langle -1, 1\rangle$
$\langle 2, 4\rangle$
$\langle 2, 2\rangle$



$[-1] = \{-1, 1, 3\}$
$= [1] = [3]$
$[2] = \{2, 4\}$

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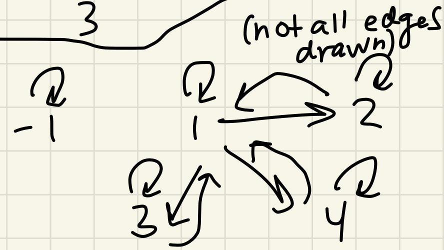
same sign

$\langle 1, 4\rangle$, ...



$[-1] = \{-1\}$
$[3] = \{3, 1, 2, 4\}$
$= [2] = [1] = [4]$

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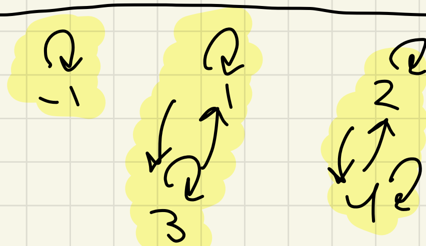
same sign and same parity



$[-1] = \{-1\}$
$[3] = \{3, 1\} = [1]$
$[2] = \{2, 4\} = [4]$

<u>Def</u> For an equivalence relation R on set A, the <u>equivalence class</u> of $a \in A$ is

$$[a] \equiv [a]_R \text{ is } \{x \in A : x R a\}$$
$$= \{x \in A : a R x\}$$

<u>Q</u>: can you give some relations (on $\mathbb{Z}$ or A) that are <u>not</u> equivalence relations?

<u>Thm</u> Let R be an equiv. rel. on set A. Let $a, b \in A$. Then $[a] = [b] \Leftrightarrow a R b$.

<u>ex</u> $[n] = [m] \Rightarrow n M_5 m$, so $n \bmod 5 = m \bmod 5$
$n M_5 m \Rightarrow [n] = [m]$

Pf ($\Rightarrow$) Assume $[a]=[b]$. WTS aRb.

$b \in [b]$ — bc R is reflexive

$b \in [a]$ — by assumption

$b \in \{x \in A : xRa\}$ — def. of equiv. class of a $[a]$

bRa — $b \in$

($\Leftarrow$) Assume aRb. WTS $[a]=[b]$. recall: To show two sets X, Y equal, $X \subseteq Y$ and $Y \subseteq X$.

We show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

$[a] \subseteq [b]$: WTS if $c \in [a]$, then $c \in [b]$.
assume $c \in [a]$.

cRa — def. of $[a]$

$$\begin{tikzcd} c \arrow[r] \arrow[rr, bend left] & a \arrow[r] & b \end{tikzcd}$$

cRb — R is transitive

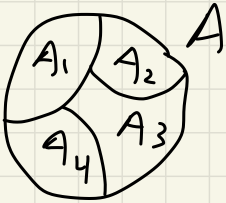
$c \in [b]$ — def. of $[b]$

Note proof of $[b] \subseteq [a]$ is symmetric. D

Def A partition of a set A is a set of non-empty subsets of A, $\{A_1, A_2, \dots, A_k\}$ such that:

1. every elt x of A is in ≥ 1 of the A_i's
2. every elt x of A is in ≤ 1 of the A_i's

$\Leftrightarrow$ every elt is in 1 of the A_i's



ex: partition $\mathbb{Z}$ into evens, odds

partition $\mathbb{Z}$ into positive, nonpositive

<u>Thm</u> Let R be an equivalence relation on set A. Then the equivalence classes partition A.

$\{[a] : a \in A\}$ is a partition of A.

(that is, every $a \in A$ is

1. in ≥ 1 equivalence class
2. in ≤ 1 equivalence class.)

<u>Proof</u> We show $\overset{①}{\geq 1}$, $\overset{②}{\leq 1}$ separately.

① WTS $\forall a \in A$: a in ≥ 1 equiv. class.

Let $a \in A$. By reflexivity, xRx $\forall x \in A$, so aRa, so $a \in [a]$.

② WTS $\forall a \in A$: a in ≤ 1 equiv. class.
That is, if $[a] \neq [c] \Rightarrow \nexists\, b \in [a] \cap [c]$.

Consider the contrapositive:

$$\exists\, b \in [a] \cap [c] \Rightarrow \underline{[a] = [c]}.$$

$$\frac{a \rightarrow b \rightarrow c}{aRc}$$

Suppose $b \in [a] \cap [c]$. WTS $[a] = [c]$.

$\underline{bRa} \wedge \underline{bRc}$ bc $b \in [a]$ and $b \in [c]$

aRb	R is reflexive
aRc	by transitivity and aRb and bRc
$[a] = [c]$	def. of equiv. class

$\square$