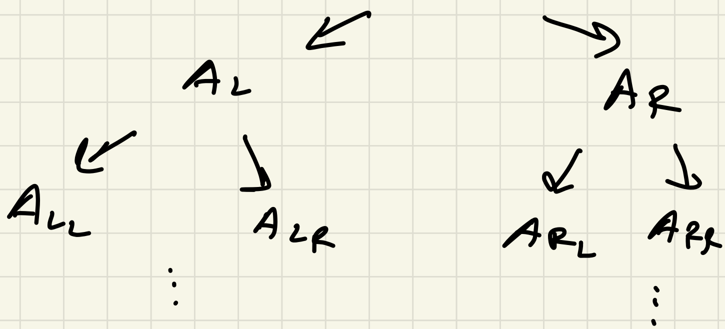
__Recursion__ is often used in CS.

- take a problem instance
- solve its subproblems
- until they are small enough

__ex__ binary search $A = \langle a_1, a_2, \dots, a_n \rangle$ sorted



until we get to the base case.

__Mathematical induction__ is a proof technique that is analogous to recursion.

__ex__ to prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$,

we prove that it works for $n=0$ (base case), and that if it holds for some $n \geq 1$, then it holds for $n+1$.

Def Let P be a predicate concerning ints ≥ 0. To give a proof by mathematical induction that $\forall n \in \mathbb{Z}^{\geq 0} \cdot P(n)$, we prove 2 things:

1) Base case: prove $P(0)$.

2) Inductive case: $\forall n \geq 1$, prove $P(n-1) \Rightarrow P(n)$

If we do (1) and (2), we've proved that $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$. why?

ex Suppose we have proven $P(0)$ and $P(n-1) \Rightarrow P(n)$. These establish $P(3)$

proof WTS $P(3)$

we know $P(0)$	base case
$P(0) \Rightarrow P(1)$	bc $P(n-1) \Rightarrow P(n)$
$P(1)$	bc $P(0)$ (modus ponens)
$P(1) \Rightarrow P(2)$	bc $P(n-1) \Rightarrow P(n)$
$P(2)$	modus ponens
$P(2) \Rightarrow P(3)$	bc $P(n-1) \Rightarrow P(n)$
$P(3)$	

claim $\forall n \geq 0, \quad \sum_{i \in \{0,1,\dots,n\}} 2^i = 2^{n+1} - 1$

i.e., $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^n - 1$

ex

n	LHS	RHS
0	$2^0 = 1$	$2^1 - 1 = 2 - 1 = 1$
1	$2^0 + 2^1 = 1 + 2 = 3$	$2^2 - 1 = 4 - 1 = 3$
2	$2^0 + 2^1 + 2^2 = 7$	$2^3 - 1 = 8 - 1 = 7$

Proof we show by induction that $\forall n \geq 0$
$$\sum_{i \in \{0,1,\dots,n\}} 2^i = 2^{n+1} - 1.$$

① we define the predicate $P(n)$ to mean that $\sum_{i \in \{0,1,\dots,n\}} 2^i = 2^{n+1} - 1$.

$$\underset{n \geq 0}{P(n)} = \begin{cases} T & \text{if } \sum 2^i = 2^{n+1} - 1 \\ F & \text{otherwise} \end{cases}$$

WTS $\forall n \geq 0 : P(n)$

we do this by showing
1) $P(0)$ (base case)
2) $\forall n \geq 1 : P(n-1) \Rightarrow P(n)$

② For the base case, we WTS $P(0)$.
i.e., WTS $\sum_{i \in \{0\}} 2^i = 2^{0+1} - 1$.

consider when $n=0$. $2^0 = 1$ and $2^{0+1} - 1 = 1$, so $P(0)$ holds.

③ For the inductive case, we need to prove $\forall n \geq 1\ P(n-1) \Rightarrow P(n)$.

$\underbrace{\text{assume } P(n-1)}_{\text{inductive hypothesis (IH)}}$. WTS $P(n)$. $\rightarrow \underbrace{\sum_{i=0}^{n} 2^i}_{\text{LHS}} = \underbrace{2^{n+1} - 1}_{\text{RHS}}$

$$\sum_{i=0}^{n-1} 2^i = 2^{(n-1)+1} - 1 \qquad (*)$$

$$\text{LHS} = \sum_{i=0}^{n} 2^i = \left[\sum_{i=0}^{n-1} 2^i\right] + 2^n \qquad \text{def. of summations}$$

$$= \left[2^{(n-1)+1} - 1\right] + 2^n \qquad \text{subs. w/ } (*)$$

$$= 2^n - 1 + 2^n \qquad \text{algebra}$$

$$= 2 \cdot 2^n - 1$$

$$= 2^{n+1} - 1 = \text{RHS}$$

So we have shown $P(n)$.

we've shown $P(0)$ and $P(n-1) \Rightarrow P(n)$, so by the principle of mathematical induction, $P(n)$ holds $\forall n \in \mathbb{Z}^{\geq 0}$.

<u>Q</u>: what would a direct proof look like?

<u>A</u>: let $X = \sum_{i=0}^{n}$.

$$x = 2x - x$$

algebra

$$= 2 \cdot \sum_{i=0}^{n} 2^i - \sum_{i=0}^{n} 2^i$$

$$= 2\left(2^0 + 2^1 + \cdots + 2^n\right) - \left(2^0 + 2^1 + \cdots + 2^n\right)$$

$$= 2 \cdot 2^0 + 2 \cdot 2^1 + \cdots + 2 \cdot 2^n - 2^0 - 2^1 - \cdots - 2^n$$

$$= \cancel{2^1} + \cancel{2^2} + \cdots + \cancel{2^n} + 2^{n+1} - 2^0 - \cancel{2^1} - \cdots - \cancel{2^n}$$

$$= 2^{n+1} - 2^0$$

$$= 2^{n+1} - 1$$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

$\square$

<u>Invalid proof:</u>

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \quad \leftarrow \text{assumed what we're trying to prove!}$$

$$\vdots$$

claim $\forall n \geq 0, \quad \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

$$0+1+2+\cdots+n = \frac{n(n+1)}{2}$$

ex

n	$0+1+2+\cdots+n$	$\frac{n(n+1)}{2}$
1	$0+1=1$	$\frac{1(2)}{2}=1$
2	$0+1+2=3$	$\frac{2(3)}{2}=3$
5	$0+1+2+3+4+5$ $=15$	$\frac{5(6)}{2}=\frac{30}{2}=15$

Proof

① we define the predicate $P(n)$ to be $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

We prove that $\forall n \geq 0 : P(n)$ using mathematical induction on n.

② For the base case, consider $n=0$. Then $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$, so

$P(0)$ holds.

(3) For the inductive case, we prove that $\forall n \geq 1$, $P(n-1) \Rightarrow P(n)$.

Let $n \geq 1$. Assume $P(n-1)$. WTS $P(n)$.

$$\sum_{i=0}^{n} i = \sum_{i=0}^{n-1} + n \qquad \text{algebra}$$

$$= \frac{(n-1)((n-1)+1)}{2} + n \qquad \text{subs.}$$

$$= \frac{(n-1)(n)}{2} + n \qquad \text{algebra}$$

$$= \frac{n(n-1)}{2} + \frac{2n}{2}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n(n-1+2)}{2}$$

$$= \frac{n(n+1)}{2}$$

$\square$

Steps to prove a "$\forall n \geq 0$, ____" statement using mathematical induction:

1. Clearly state the property $P(n)$ and that you are proving using induction. Be sure to state what variable you are performing induction over.

2. Prove $P(0)$ (base case)

3. Prove $\forall n \geq 1$, $P(n-1) \Rightarrow P(n)$.