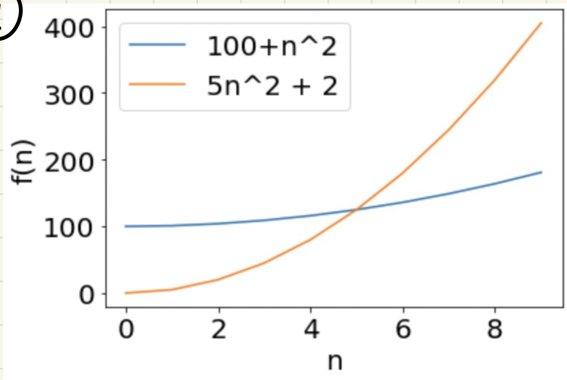
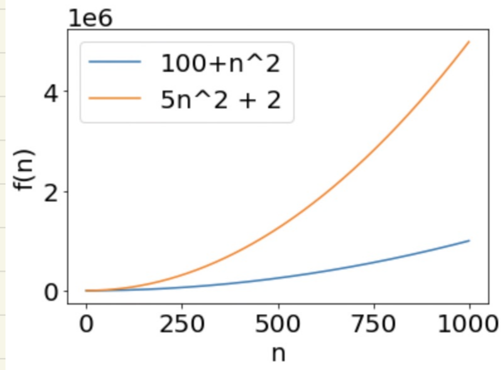
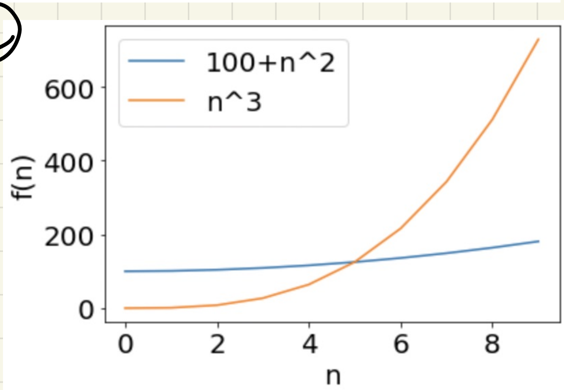
which function is "smaller"?

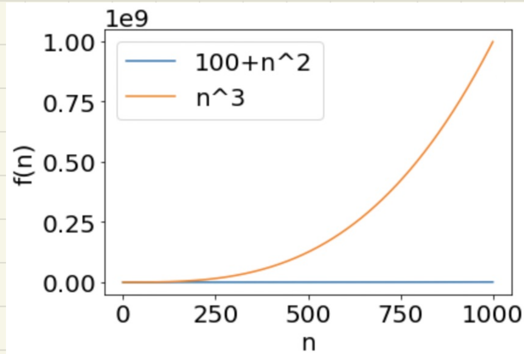
①





②





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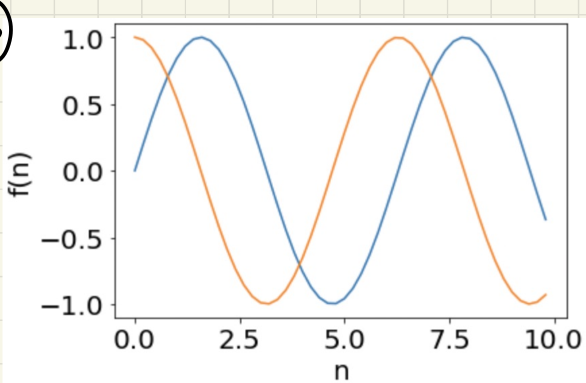


In CS, we focus on "grows no faster than" as an approximation of "smaller than."

<u>Def</u>. $f: \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $g: \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$.

we say $f = O(g)$ "f is big O of g" if

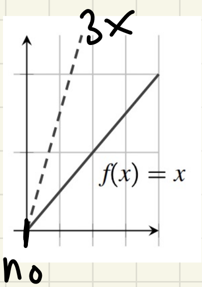
$\exists\, c>0,\ n_0 \geq 0$ s.t. $\forall n \geq n_0 : f(n) \leq c \cdot g(n)$.

Note: $f = O(g)$ is standard notation, but it uses "=" to mean "has the property."

To prove $f(n) = O(g(n))$, we need to construct n_0, c s.t. $\forall n \geq n_0 : f(n) \leq c \cdot g(n)$

To prove $f(n) \neq O(g(n))$, we need to show that $\forall n_0 \geq 0,\ c>0 : \exists n \geq n_0 : f(n) > c \cdot g(n)$.

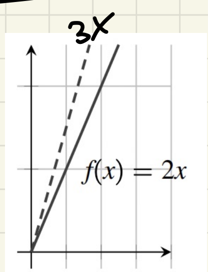
Examples: all of the following functions f are $O(n)$.



WTS $f(n) = n$ is $O(n)$.
Let $n_0 = 0,\ c = 3$.

$\forall n \geq 0 : f(n) = n \leq 3n$

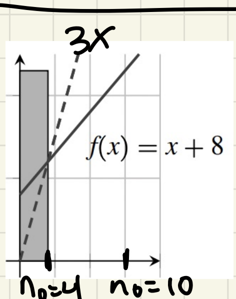
$\square$



WTS $f(n) = 2n$ is $O(n)$.

Let $n_0 = 0,\ c = 3$.

$\forall n \geq 0 : 2n \leq 3n$ $\square$



WTS $f(n) = x + 8$ is $O(n)$.

Suppose we want to use $c = 3$.
What n_0 can we choose?

could just guess + check: how about $n_0 = 10$?

$3n = 30$ when $n = 10$
$n + 8 = 18$ when $n = 10$

so $n_0 = 10$ works.

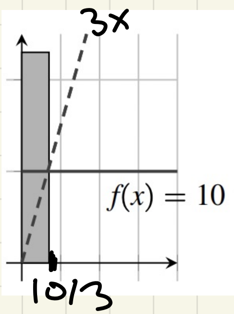
smallest n_0? plug in: $n_0 + 8 = 3n_0$

$8 = 2n_0$

$4 = n_0$

any $n_0 \geq 4$ would work.

let $n_0 = 4$, $c = 3$. $\forall n \geq n_0 : n + 8 \leq 3n$. $\square$

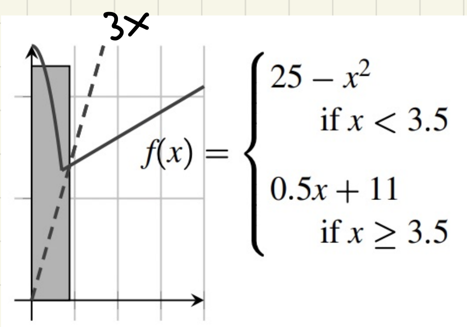


WTS $f(n) = 10$ is $O(n)$.

let $n_0 = 3.4$, $c = 3$.

$\forall n \geq 3.4 : 10 \leq 3n$

$\square$



WTS $f(n) = O(n)$.

Does $n_0 = 4$, $c = 3$ work?

we would need $\forall n \geq 4$:

$0.5n + 11 \leq 3n$.

but when $n = 4$, $0.5n + 11 = 13$
and $3n = 12$. X

$13.5 \leq 15$

let $n_0 = 5$, $c = 3$. $\forall n \geq n_0 : f(n) \leq c \cdot n$.

Example: $n^3 \neq O(n^2)$.

Proof: WTS $\forall c > 0, n_0 \geq 0 : \exists n \geq n_0 : n^3 > c n^2$

we prove by showing how to construct n for any c, n_0.

let $c > 0$ and $n_0 \geq 0$. we need $n \geq n_0$ s.t. $n^3 > c \cdot n^2$. let $n = (c+1)$. Then $n^3 = (c+1)^3$ and $c \cdot n = c(c+1)^2$, so $n^3 > c \cdot n^2$.

we have n s.t. $n^3 > cn^2$, but also need $n \geq n_0$. let $n = \max\{n_0, c+1\}$. Now $n \geq n_0$ and $n^3 > cn^2$. $\square$

If $n_0 = 100$ and $c = 5$, $n = 6$ would not be a valid n.

Logarithms for positive real number $b \neq 1$ and real number $x > 0$, $\log_b x$ is the real number y s.t. $b^y = x$.

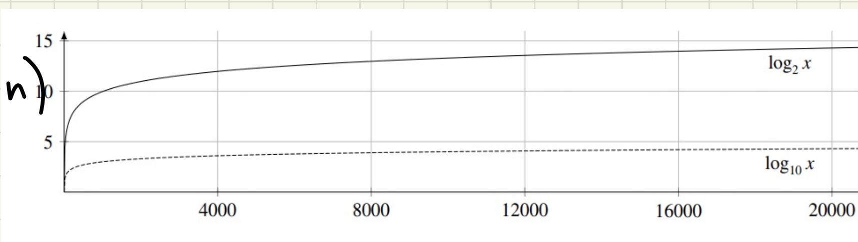
$\log_4 16$ means "the number we need to raise 4 to to get 16"

Lemma 6.7 let $b > 1$ and $k \geq 0$.

$\log_b(n^k) = O(\log n)$. Base, exponents in logs don't matter asymptotically.

ex

$\log_{10} n = O(\log_2 n)$



Proof WTS $\exists c > 0, n_0 \geq 0$ s.t. $\forall n \geq n_0$:
$\log_b(n^k) \leq c \log_a n$.

can be anything, so we will drop later

Note that

$$\log_b(n^k) = k \cdot \log_b(n)$$ by def. of logs + exponents

$$= k \frac{\log_a n}{\log_a b}$$ change of base rule $\log_b x = \frac{\log_a x}{\log_a b}$

now take $n_0 = 1$ and $c = \frac{k}{\log_a b}$.

$\forall n \geq 1$, $\log_b(n^k) = k \frac{\log_a n}{\log_a b} \leq \frac{k}{\log_a b} \log_a n$,

so $\log_b(n^k) = O(\log_a n)$. $\square$

Since a could be anything, we drop it.

Lemma let $b, d > 1$. If $d < b$ then $b^n \neq O(d^n)$.

ex $3^n \neq O(2^n)$.

This lemma tells us that the base of an exponent does matter, asymptotically.