Def A predicate is a Boolean-valued function $P: U \to \{T, F\}$ for a set U.
That is, a rule / property that a particular entity may / may not have.

ex $\text{isEven}(n) := \begin{cases} T & \text{if } n \text{ even} \\ F & \text{if } n \text{ odd} \end{cases}$, $n \in \mathbb{Z}$

$\text{isPrime}(n) := \begin{cases} T & \text{if } n \text{ prime} \\ F & \text{if } n \text{ not prime (composite)} \end{cases}$, $n \in \mathbb{Z}^{>0}$

$\text{isSubset}(A, B) := \begin{cases} T & \text{if } A \subseteq B \\ F & \text{if } A \not\subseteq B \end{cases}$

$\text{isRat}(x, y) := \begin{cases} T & \text{if } x/y \in \mathbb{Q} \\ F & \text{if } x/y \notin \mathbb{Q} \end{cases}$

On its own, a predicate $P(x)$ has no truth value. The value x is unbound.

Can make it a proposition by applying it to a specific entity.

ex $\text{isEven}(n)$ ✗ $\quad$ $\text{isEven}(1)$ ✓

$\text{isEven}(2)$ ✓

$\text{isPrime}(n)$ ✗ $\quad$ $\text{isPrime}(2)$ ✓

Can also use quantifiers:

ex there exists int n s.t. $\text{isEven}(n)$ ✓ T

for all int n $\text{isEven}(n)$ ✓ F

Universal Quantifier $\forall$ "for all"

$\forall x \in S : P(x)$

"for all x in S, $P(x)$ is true" ✓

true iff $P(x)$ evaluates to T for every $x \in S$

Existential Quantifier $\exists$ "there exists"

$\exists x \in S : P(x)$

"there exists x in S s.t. $P(x)$ is true"

true iff $P(x)$ evaluates to T for some $x \in S$

These quantifiers $\forall x$ or $\exists x$ <u>bind</u> x.

<u>ex</u>

$\forall x \in \mathbb{Z}$	: x is even	F	
$\forall x \in \mathbb{Z}$	: $2x$ is even	T	
$\forall n \in \mathbb{Z}$	: if n^2 even then n even	T	
$\forall x \in \mathbb{Z}$	: if $7x+9$ even then x odd	T	
$\forall x, y \in \mathbb{R}$	: if x/y irrational then x or y irrational	T	
$\forall A, B$	: $(A \subseteq B) \Rightarrow (A - B) = \emptyset$	T	
$\exists n \in \mathbb{Z}$	: n is not even	T	$(n=3)$
$\exists n \in \mathbb{Z}$	: n is prime	T	$(n=3)$
$\exists n \in \mathbb{Z}$	: n is even and n is odd	F	
$\exists x, y \in \mathbb{Q}$	: $x \cdot y \in \mathbb{Q} \wedge \neg(x, y \in \mathbb{Q})$	T	

Precedence:

- $\forall, \exists$ highest
- use ( ) to override or for clarity

<u>ex</u> $(\forall x \in S : P(x)) \Rightarrow (\exists y \in S : P(y))$

vs. $\forall x \in S : [P(x) \Rightarrow \exists y \in S : P(y))$

We have to be careful when going between English language and quantified statements.

ex All students do not pay full tuition.

an attempt: $\forall$ students x : x does not pay full tuition

but this means that no student pays full tuition - not what we meant!

we want "not all students pay full tuition"

$\neg$($\forall$ students x : x does not pay full tuition)

ex All ints are not even $\forall x \in \mathbb{Z} : \neg(\text{isEven}(x))$
Not all ints are even $\neg(\forall x \in \mathbb{Z} : \text{isEven}(x))$

≥ 1 of ints x and y are even.
$\equiv \text{isEven}(x) \vee \text{isEven}(y)$

x is even but y is odd
$\equiv \text{isEven}(x) \wedge \neg\text{isEven}(y)$

Unbound Variables

ex n^2 even $\Rightarrow n$ even — true $\forall n$
n even $\Rightarrow 6 | n$ — false for some n

$\forall n : n \text{ even} \Rightarrow 6 | n$ F
$\exists n : n \text{ even} \Rightarrow 6 | n$ T

If a quantifier is missing, we assume $\forall$.

ex $3 | x$ becomes $\forall x : 3 | x$

Def An expression is <u>fully quantified</u> if it has no unbound variables.
(note that we can assume $\forall$ if missing)

ex all primes are odd

$$\forall p \in Primes : isodd(p)$$

$$\forall p \in \mathbb{Z} : [isPrime(p) \Rightarrow isodd(p)]$$

$\hookrightarrow$ nope! we can disprove w/ $p=2$, i.e.,
$$\exists p \in Primes : \neg isodd(p)$$

So in general:

$\forall x \in S : P(x)$ is disproved by showing $\exists x \in S : \neg P(x)$.