Randomness + probability uses in CS:

- randomized algorithms
- data structures using randomness
- modeling real-world phenomena

But first, we need to learn to count!

Sum rule: If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

Product rule: The number of pairs $\langle x, y \rangle$ with $x \in A$, $y \in B$ is $|A| \cdot |B|$.

$$|A \times B| = |A| \cdot |B|$$

ex A restaurant has 2 lunch specials.
1. soup or salad
2. soup and salad

If A = set of soups = {chicken noodle, tomato, ...}

B = set of salads = {caesar, cobb, house...}

How many possibilities are there for 1 and 2?

1 : $|A| + |B|$

2 : $|A| \cdot |B|$

More general product rule:

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot |A_3| \cdot \cdots \cdot |A_k|$$

ex How many 32-bit strings are there?

$\underbrace{010\cdots001}_{\text{32-bit string}}$

$$|\{0,1\}^{32}| = \underbrace{|\{0,1\}| \cdot |\{0,1\}| \cdots |\{0,1\}|}_{\text{32 times}} = 2^{32}$$

Def Given some random process, the sample space S is the set of all possible outcomes.

↗ also called prob. distribution

A probability function $Pr: S \to \mathbb{R}$ describes the fraction of the time that $s \in S$ occurs.

$$\sum_{s \in S} Pr[s] = 1$$

$$Pr[s] \geq 0 \ \forall s \in S$$

ex

- flipping a coin

$S = \{\text{heads}, \text{tails}\}$

$Pr[\text{heads}] = 0.5$
$Pr[\text{tails}] = 0.5$

$$\sum_{s \in S} Pr[s] = 0.5 + 0.5 = 1$$

- drawing a card

$S = \{ 2 \text{ clubs}, 3 \text{ clubs}, \dots \}$

$Pr[s] = \frac{1}{52} \ \forall s \in S$

- flipping 2 coins

$S = \{ \langle H, H \rangle, \langle H, T \rangle, \langle T, H \rangle, \langle T, T \rangle \}$

each has probability 0.25

all of these have <u>uniform</u> probability. But probability functions can be non-uniform.

<u>ex</u> Let $S = \{ 0, 1, 2, \dots, 7 \}$. Choose from S by flipping 7 coins and counting # of H.

HHHHHHH $\to$ 7 $\quad$ $Pr[4] \approx 0.2734$

$\vdots$ $\quad$ $Pr[7] \approx 0.0078$

<u>Def</u> A set of outcomes is called an <u>event</u>.

$$E \subseteq S, \quad Pr[E] = \sum_{s \in E} Pr[s]$$

<u>ex</u>

when flipping 2 coins, the probability that at least one is H is $0.25 + 0.25 + 0.25 = 0.75$.

when drawing 1 card, the probability that it is an ace is $4/52 = 1/13$.

Theorem 10.4: properties of event probs.

Let S be a sample space and $A \subseteq S$, $B \subseteq S$ events. Let $\bar{A} = S - A$ be the complement of event A.

$Pr[S] = 1$

$Pr[\emptyset] = 0$

$Pr[\bar{A}] = 1 - Pr[A]$

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

ex

- when drawing 1 card, what is the probability that it's not an ace?

$S = \{$ all cards $\}$

$A = \{$ A clubs, A spades, A hearts, A diamonds $\}$

$Pr[\bar{A}] = 1 - Pr[A] = 1 - 0.75 = 0.25$

- when drawing 1 card, what's the prob. that it's a Q or a heart?

$A = \{$ Q clubs, Q hearts, Q diamonds, Q spades $\}$

$B = \{$ all cards heart $\}$

$A \cup B = \{$ all hearts + 3 queens $\}$

$A \cap B = \{$ Q hearts $\}$

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

$= 4/52 + 13/52 - 1/52 = 16/52$