Review

Let A, B be sets. $f: A \to B$ is a function if it satisfies these props:

1) $\forall a \in A$, $f(a)$ is defined

2) $\forall a \in A$, $f(a)$ does not produce 2 diff. outputs

3) $\forall a \in A$, $f(a) \in B$.

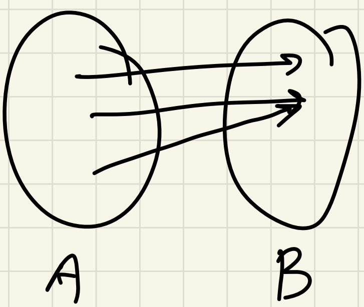
A domain
B codomain
$\{f(a) : a \in A\}$ range

examples:

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(\langle x, y \rangle) = x + y$

$E: \mathbb{Z} \to \{T, F\}$ $\quad E(x) = x$ is even



$a \in A$	$b \in B$
a_1	$f(a_1)$
a_2	$f(a_2)$
a_3	$f(a_3)$
$\vdots$	

all elts of A have exactly 1 row

doesn't have to have all elts of B; dupes OK

ex $S: \mathbb{Z} \to \mathbb{Z}$ defined by $S(x) = x+1$
(successor function)

domain: $\mathbb{Z}$
codomain: $\mathbb{Z}$
range: $\mathbb{Z}$

claim: $S: \mathbb{Z} \to \mathbb{Z}$ is a function.

proof: we prove all 3 properties.

1) $\forall x \in \mathbb{Z}$, $s(x)$ is defined as $x+1$.

2) To show that $\forall x \in \mathbb{Z}$, $S(x)$ does not produce 2 diff outputs, we show that if $s(x) = a$ and $s(x) = b$, then $a = b$.

suppose $S(x) = a$ and $S(x) = b$.

$a = x+1$, $b = x+1$ — def. of S

$a = b$ — substitution

3) $\forall x \in \mathbb{Z}$, $S(x) = x+1$ is an integer because the sum of integers is integer.

ex $f: \mathbb{R}^{\geq 0} \to \mathbb{R}$ defined by $f(x) =$ the number whose absolute value is x

Is f a function?

no! violates prop. 2

Def A function $f: A \to B$ is

1. onto (surjective) if

$\forall b \in B \ \exists a \in A : f(a) = b$

$\equiv \forall b \in B$, something in A maps to it

$\equiv \forall b \in B$, b shows up in ≥ 1 row of table

$\equiv$ Codomain = range

2. one-to-one (injective) if
   1:1

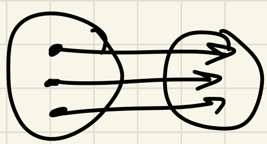
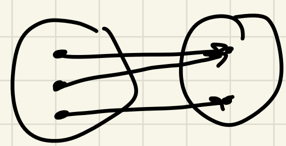
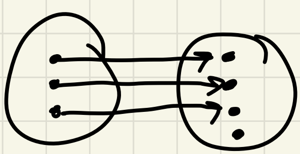
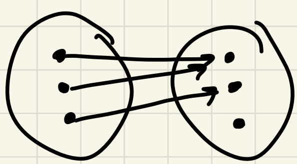
$\forall a_1, a_2 \in A \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

$\equiv \forall b \in B$, at most 1 thing in A maps to it

$\equiv \forall b \in B$, b shows up in ≤ 1 row of table

3. a bijection if both onto and one-to-one

$\forall b \in B$, exactly 1 elt of A maps to it

	1:1	not 1:1
onto		
not onto		

How to prove that f is onto / 1:1

<u>onto:</u>

WTS $\forall b \in B \; \exists a \in A : f(a) = b$

$\equiv$ if $b \in B$ then $\exists a \in A : f(a) = b$.

Step 1: Suppose $b \in B$.

Step 2: Show that $\exists a \in A : f(a) = b$ by <u>constructing</u> a s.t. $f(a) = b$.

<u>ex</u> recall $S : \mathbb{Z} \to \mathbb{Z}$, $S(x) = x+1$.

<u>claim:</u> S is onto.

$S(x) = x+1$. so $S(b-1) = b$.

<u>proof</u> let $b \in \mathbb{Z}$. we need to show that $\exists a \in \mathbb{Z} : S(a) = b$. Consider $a = b-1$. $a \in \mathbb{Z}$, and $S(a) = b-1+1 = 1$, as needed. $\square$

(this is an example of proof by construction)

<u>not onto:</u>

WTS $\neg(\forall b \in B \; \exists a \in A : f(a) = b)$

$\equiv \exists b \in B \; \forall a \in A : f(a) \neq b$

construct a $b \in B$ s.t. nothing in A maps to it.

ex $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ not onto

proof consider $b = -1 \in \mathbb{R}$.

$\forall a \in \mathbb{R} : f(a) = a^2$ def. of f

$\forall a \in \mathbb{R} : f(a) \geq 0$ property of ^{2}

$\forall a \in \mathbb{R} : f(a) \neq b$ $b < 0$

invalid proof that f is onto:

let $b \in \mathbb{R}$. WTS $\exists a \in \mathbb{R} : f(a) = b$. Consider $a = \sqrt{b}$. Since $\underbrace{b \in \mathbb{R}, \sqrt{b} \in \mathbb{R}}_{\text{false}}$. Also, $f(a) = \sqrt{b}^2 = b$. $\square$

1:1

WTS $\forall a_1, a_2 \in A$ $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

direct proof:

1. suppose $a_1, a_2 \in A$ and $a_1 \neq a_2$
2. show that $f(a_1) \neq f(a_2)$

contrapositive proof:

1. Suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$
2. Show that $a_1 = a_2$

this is often how we prove 1:1

ex S is 1:1

Pf Suppose $a_1, a_2 \in \mathbb{Z}$ and $S(a_1) = S(a_2)$.

$a_1 + 1 = a_2 + 1$ def of S

$a_1 = a_2$ algebra $\square$

ex $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $f(x) = \frac{1}{x} + 1$ is 1:1

Pf Suppose $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $f(a_1) = f(a_2)$

$\frac{1}{a_1} + 1 = \frac{1}{a_2} + 1$ def of f

$\frac{1}{a_1} = \frac{1}{a_2}$ algebra

$a_1 = a_2$ algebra $\square$

not 1:1

$\neg[\forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)]$

$\equiv \exists a_1, a_2 \in A : \neg(a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$

$\equiv \exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$

(disproof by counterexample)

ex $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ not 1:1

let $a_1 = 2 \in \mathbb{R}$ $\qquad f(a_1) = 2^2 = 4$

$a_2 = -2 \in \mathbb{R}$ $\qquad f(a_2) = (-2)^2 = 4$

so $a_1 \neq a_2$ and $f(a_1) = f(a_2)$

Note: to prove f is a bijection, prove both:

f is onto

f is 1:1

we did this for $S(x) = x+1$