<u>Def</u> A <u>path</u> in $G=(V,E)$ is a sequence of nodes $(u_1, u_2, \dots, u_k)$ s.t.

- $\forall i \in \{1,2,\dots,k\} : u_i \in V$ $\qquad u_1 \to u_2 \to \cdots \to u_k$
- $\forall i \in \{1,2,\dots k-1\}$:

  $\langle u_i, u_{i+1} \rangle \in E$ (or $\{u_i, u_{i+1}\} \in E$ if undirected)

<u>Q</u> does this definition allow repeated nodes in a path?

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] \arrow[l, bend left] & d \end{tikzcd}$$

$u_1 = a$
$u_2 = b$
$u_3 = c$
$u_4 = b$
$u_5 = c$
$u_6 = d$

A path is <u>simple</u> if all of its nodes are distinct.

The <u>length of a path</u> is # edges $k-1$.
We say a path <u>traverses</u> its edges.

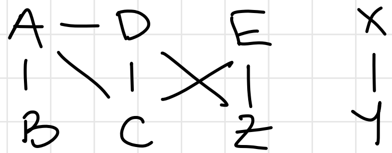
The <u>shortest path</u> from node u to node v is the path of minimum length from u to v.

$d(u,v)$ $\quad dist(u,v)$

The <u>distance</u> between nodes u, v is the length of the shortest path between u, v.

A graph is <u>connected</u> if $\forall u,v \in V$ $\exists$ a path from u to v.

ex



between A, D :

AD	1
ACD	2
ACEZD	4

dist(A,D) = 1

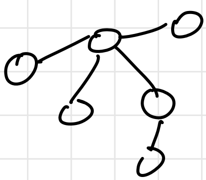
Def A cycle $(u_1, u_2, \dots, u_k, u_1)$ is a path of length ≥ 2 from u_1 to u_1 that does not traverse the same node twice.

A cycle is simple if each node is distinct.

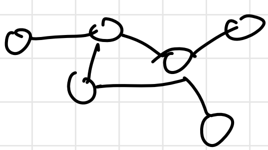
A graph is acyclic if it contains no cycles.

ex

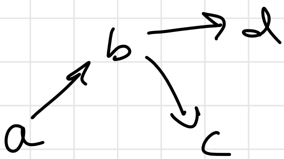
acyclic



acyclic



not acyclic



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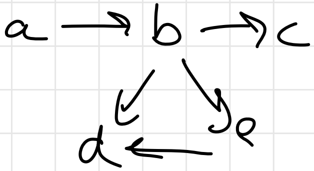
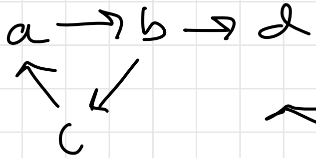


acyclic



not acyclic

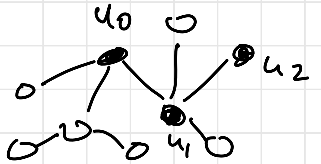
How many cycles does have ?

$\boxed{1}$ or 3 ? we consider $\langle a, b, c \rangle$, $\langle b, c, a \rangle$, $\langle c, a, b \rangle$ to be all same cycle.

<u>lemma 4.33</u> If $G = (V, E)$ is acyclic then $\exists\, v \in V$ s.t. $\deg(v) = 0$ or $\deg(v) = 1$.

<u>Pf</u> we give a proof by construction via an algorithm that, given any acyclic graph, finds a degree 0 or degree 1 node.

alg: let u_0 = any node in G
let $i = 0$
while current node u_i has unvisited neighbors:
  let u_{i+1} = such a neighbor
  $i = i+1$
return u_i

(example: )

Given an acyclic graph G, let t be the node returned by the alg.

There are 2 cases:

Case 1: $t = u_0$ $\quad \bullet\, u_0$ $\quad$ so $\deg(t) = 0$

So G has a node of degree 0, t.

Case 2: $t = u_k$, $k \geq 1$. $\quad$ $u_0 — u_1 \cdots \overset{t}{u_k}$

We WTS $\deg(t) = 1$.

Since t is last in the sequence $\langle u_0, u_1, \dots, u_k \rangle$ there is no edge from t to any unvisited node.

If $\exists$ edge from t to any node other than u_{k-1} then it is some node in $\{u_0, u_1, \dots, u_{k-2}\}$.

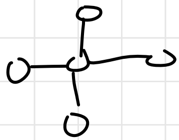
$$u_0 — u_1 — \cdots\ u_j\ \cdots — u_{k-1} — u_k$$

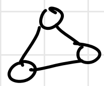
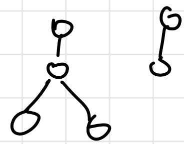
But then $\langle u_j, \dots, u_k, u_j \rangle$ is a cycle, contradicting that G is acyclic.

So t has only one edge, the edge to u_{k-1}.
So $\deg(t) = 1$.

Since the cases are exhaustive, the claim is proved.

Def A tree is a undirected graph that is connected and acyclic.

ex  ✓  ✓  ✓

 ✗  ✗ but is a forest (undirected, acyclic graph)

Thm 11.35 Let $T = (V, E)$. Then $|E| = |V| - 1$.

Pf : in book. (by mathematical induction)

Corollary 11.36 Let $T = (V, E)$. Then

1. adding any edge $e \in E$ to T creates a cycle
2. removing any edge $e \in E$ from T disconnects the graph