Note:

$f: A \to B$ is onto $\Rightarrow$ $|A| \geq |B|$

$f: A \to B$ is 1:1 $\Rightarrow$ $|A| \leq |B|$ $\star$

$f: A \to B$ is a bijection $\Rightarrow$ $|A| = |B|$

recall defs:

f onto: $\forall b \in B \; \exists a \in A : f(a) = B$

f 1:1 : $\forall a, a' \in A \quad a \neq a' \Rightarrow f(a) \neq f(a')$

Theorem 9.13 (Pigeonhole Principle, PHP)

Let A, B be sets and $f: A \to B$ a function.

If $|A| > |B|$ then there are 2 distinct $a, a' \in A$ s.t. $f(a) = f(a')$.

$|A| > |B| \Rightarrow \exists a, a' \in A : (a \neq a') \wedge (f(a) = f(a'))$

Proof The PHP is the contrapositive of $\star$

The "pigeon" way of thinking of the PHP:

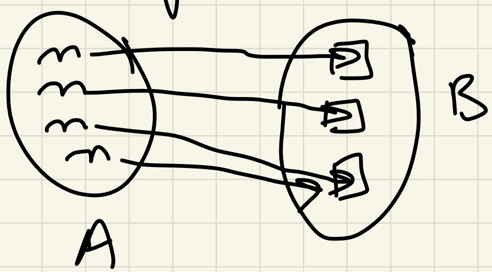
suppose you have $A = n+1$ pigeons

$B = n$ pigeon cubbies

each pigeon flies to a cubby.

f(pigeon a) = pigeon a's cubby.

≥ 2 pigeons share a cubby.



<u>claim</u> among 13 people, ≥ 2 share a birth month.

<u>Pf</u> let A (pigeons) be the set of 13 people

let B (pigeonholes) be the set of 12 months.

$f: A \to B$ defined by f(person a) = a's birth month.

is f a function?

1) each a has a birth month
2) each a has just 1 birth month
3) each birth month is in set of 12 months

Note that $\left.\begin{array}{l} |A| = 13 \\ |B| = 12 \end{array}\right\} \Rightarrow |A| < |B|$

Thus by PHP $\exists a_1, a_2 \in A$ s.t. $a \neq a_1$ and $f(a_1) = f(a_2)$.

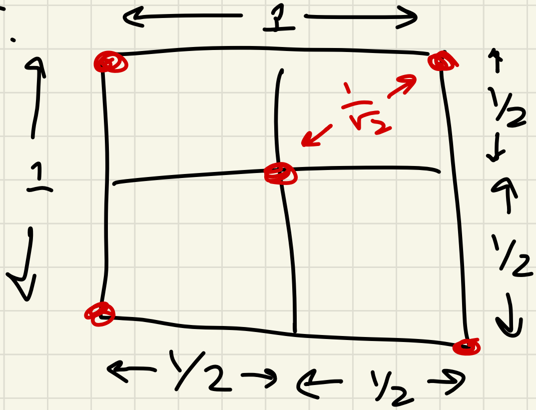
so there 2 distinct people a_1, a_2 s.t. a_1's and a_2's birth months are same.

claim (9.36) Let $n \geq 0$, integer.

Suppose $\exists\, n^2+1$ points in the unit square. Then $\exists$ 2 points within $\sqrt{2}/n$ of each other.

ex $n=2$, so $n^2+1=5$.

2 of these points are within $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ of each other



Pf Let A be the set of n^2+1 points.

$a^2+b^2=c^2$

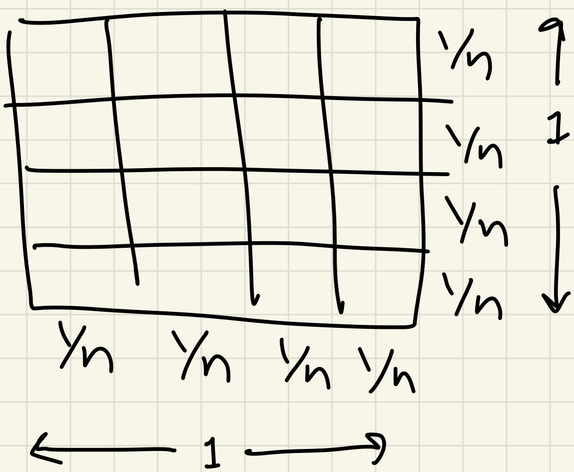
$\frac{1}{2}^2+\frac{1}{2}^2=c^2$

$\frac{1}{4}+\frac{1}{4}=c^2$

$\frac{1}{2}=c^2$

$\frac{1}{\sqrt{2}}=c$

Let B be the set of n^2 $1/n \times 1/n$ subsquares of the unit square.



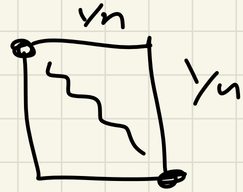
Let f(point a) = the $1/n \times 1/n$ subsquare that contains a.

Is f well-defined? Check 1, 2, 3. ↑ choose something for boundary e.g., below + left.

Note $\left.\begin{array}{l}|A| = n^2+1 \\ |B| = n^2\end{array}\right\} \Rightarrow |A| > |B|$

By PHP, $\exists\, a_1, a_2 \in A$ s.t. $a_1 \neq a_2$, $f(a_1) = f(a_2)$, i.e., $\exists$ 2 distinct points a_1, a_2 s.t. a_1, a_2 are within some $1/n \times 1/n$ subsquare.

Within a $1/n \times 1/n$ square, the farthest 2 points can be is $\frac{\sqrt{2}}{n}$:



$$\sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2}$$
$$= \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n}.$$

So a_1, a_2 are within $\frac{\sqrt{2}}{n}$ of each other.

$\square$