A special compound proposition: $\neg q \Rightarrow \neg p$

The <u>contrapositive</u> of $p \Rightarrow q$.

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$q \Rightarrow p$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

special because $\quad p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

Note that the converse of $p \Rightarrow q$, $q \Rightarrow p$, is not logically equivalent to $p \Rightarrow q$.

$$p \Rightarrow q \not\equiv q \Rightarrow p$$

Recall proofs by contradiction:

Claim 4.18 (part of it)

If $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$. $\quad p \Rightarrow q$

Why did we prove by contradiction? Let's try a direct proof.

Let n^2 be even. WTS n is even.

$n^2 = 2c$ for $c \in \mathbb{Z}$ $\quad$ def. of even

??? $\quad n = \sqrt{2c}$ $\quad$ we don't have facts about things like this.

$n = \frac{2c}{n}$

1. For contradiction, suppose $\neg(p \Rightarrow q)$
2. $\neg(p \Rightarrow q) \equiv p \wedge \neg q$
3. direct proof that $\neg q \Rightarrow \neg p$
4. established $\neg p \wedge p$
5. noted that $\neg p \wedge p \equiv F$ (a contradiction)
6. $\neg(p \Rightarrow q) \equiv F$, so $p \Rightarrow q \equiv T$.

(1)

Proof For contradiction, suppose the claim is false. That is, suppose that $\underbrace{n^2 \text{ is even}}_{p}$ (2) but $\underbrace{n \text{ is odd}}_{\neg q}$.

$$\left.\begin{array}{l} n = 2k+1 \text{ for } k \in \mathbb{Z} \;\Big)\; \neg q \\ n^2 = (2k+1)^2 \\ n^2 = 4k^2 + 4k + 1 \\ n^2 = 2(2k^2+2k)+1 \\ n^2 = 2c+1 \text{ for } c \in \mathbb{Z} \\ n^2 \text{ is odd} \;\Big)\; \neg p \end{array}\right\}$$

n odd $\Rightarrow n^2$ odd

$\neg q \Rightarrow \neg p$

so p (by (2))
and $\neg p$ (by (3))

(4)

This contradicts that n^2 is even, so our initial assumption that n is odd is false. (5)
So the initial claim is true. (6)

Note: (3) was a direct proof that $\neg q \Rightarrow \neg p$. That is, a direct proof of the contrapositive.

So we can give a shorter proof by proving the contrapositive!

Claim If n^2 is even, then n is even.

Proof We will prove the contrapositive: if n is odd, then n^2 is odd.

$n = 2k+1$ for $k \in \mathbb{Z}$ def. of odd

$n^2 = (2k+1)^2$

$n^2 = 4k^2 + 4k + 1$

$n^2 = 2(2k^2 + 2k) + 1$

$n = 2c + 1$ for $c \in \mathbb{Z}$ $c = 2k^2 + 2k$; prod, sum of ints is int

n^2 is odd

$\square$

Note you can only use contrapositive proofs on if-then claims ($p \Rightarrow q$)
sometimes a direct proof is easier/simpler.
sometimes not.

Proposition Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then x is odd.

Proof (direct) Suppose $7x+9$ is even. WTS x is odd.

$7x + 9 = 2c$ for $c \in \mathbb{Z}$ def. of even

$x = 2c - 6x - 9$ algebra

$x =$ even − even − odd   prod. of evens w/ any int is even

$x =$ even − odd   even − even is even

x is odd   diff of even and odd is odd

$\square$

<u>Proof</u> (by contrapositive)

we prove the contrapositive. That is, if x is even then $7x+9$ is odd.
Suppose x is even. WTS $7x+9$ is odd.

$7x$ is even   prod of any int and even is even

$7x+9$ is odd   sum of even and odd is odd

$\square$

(4.17)

<u>Claim</u> Suppose $y \neq 0$. If $\underbrace{x/y \text{ is irrational}}_{p}$, then $\underbrace{x \text{ is irrational or } y \text{ is irrational}}_{(q \vee r)}$. $p \Rightarrow (q \vee r)$

contrapositive is $\neg(q \vee r) \Rightarrow \neg p \equiv (\neg q \wedge \neg r) \Rightarrow \neg p$

<u>Proof</u> we prove the contrapositive. That is, if x and y rational, then x/y rational.

You did this or HW 1 :)

claim (4.16) If $|x|+|y| \neq |x+y|$ then $xy<0$.

ex

x	y	$\|x\|+\|y\|$	$\|x+y\|$	xy	
-2	3	5	1	-6	T T
2	3	5	5	6	F F

Pf We prove the contrapositive. That is, if $xy \geq 0$ then $|x|+|y| = |x+y|$.
Suppose $xy \geq 0$. WTS $|x|+|y| = |x+y|$.

We prove using cases.

Case 1: $x, y \geq 0$.

$|x|+|y| = x+y$ by def of $| \, |$, $x \geq 0$, $y \geq 0$

$x+y = |x+y|$ bc $x, y \geq 0 \Rightarrow x+y \geq 0$, def of $| \, |$

Case 2: $x, y \leq 0$.

$|x|+|y| = -x+-y$ def. of $| \, |$, $x, y \leq 0$

$-x-y = -(x+y)$ algebra

$-(x+y) = |x+y|$ bc $x, y \leq 0 \Rightarrow x+y \leq 0$, def of $| \, |$

$\square$

<u>claim</u> All prime numbers are odd.

$\equiv$ If p is prime, then p is odd.

Disproof by counterexample: 2 is prime but 2 is not odd.

claim. Let $p \geq 3$. If p is prime, then p is odd.

<u>Pf</u> We prove the contrapositive. That is, we let $p \geq 3$, and we show that if p is even, then p is not prime.
Let $p \geq 3$ and p even.

p has 2 as a divisor	given
p is not prime	$p \neq 2$, so p has a divisor not equal to itself or 1.

$\square$