Review of predicate logic:

- for all, aka the universal quantifier:

  $\forall x \in S : P(x)$
  T iff $P(x)$ is T for every $x \in S$

- there exists, aka the existential quantifier:

  $\exists x \in S : P(x)$
  T iff $P(x)$ is T for some (≥ 1) $x \in S$.

Def A fully quantified expression in predicate logic is a theorem iff it is true for every possible meaning of its predicates. (akin to a tautology)

aka, $\forall S$

Thm (3.39) Let S be any set. $\forall x \in S : [P(x) \vee \neg P(x)]$.

note the implied $\forall P$

ex $P(x) = \text{isEven}(x)$, $S = \mathbb{Z}$.

$\forall x \in \mathbb{Z} : [\text{isEven}(x) \vee \neg \text{isEven}(x)]$

Pf For any $x \in S$, $P(x)$ is defined, and $P(x) = T$ or $P(x) = F$ — def. of predicate

for any $x \in S$, $P(x) \vee \neg P(x)$ — def. of $\vee, \neg$

$\forall x \in S : [P(x) \vee \neg P(x)]$ — def. of $\forall$

Non-thm (3.40) $[\forall x \in S : P(x)] \vee [\forall x \in S : \neg P(x)]$

note implied $\forall S, \forall P$

ex $[\forall x \in \mathbb{Z} : \text{isEven}(x)] \vee [\forall x \in \mathbb{Z} : \neg \text{isEven}(x)]$

all ints are even or all ints are odd

disproof: $x = 3$
$\exists x \in \mathbb{Z} : \neg \text{isEven}(x)$

disproof: $x = 2$
$\exists x \in \mathbb{Z} : \text{isEven}(x)$

Disproof of 3.40: we gave a $P(x)$ and an S (isEven and $\mathbb{Z}$) s.t. 3.40 is not true.

Def Fully quantified expressions φ and ψ (phi, psi) are logically equivalent ($\varphi \equiv \psi$) iff "$\psi \Leftrightarrow \varphi$" is a theorem — that is, they have the same meaning under every interpretation of predicates.

$\forall S, \forall P$

Thm (3.41) $\neg[\forall x \in S : P(x)] \Leftrightarrow [\exists x \in S : \neg P(x)]$

ex to disprove $\forall x \in \mathbb{Z} : \text{isEven}(x)$, just find $x \in \mathbb{Z} : \neg \text{isEven}(x)$. e.g., $x = 3$.

this theorem explains why disproof by counter example works!

Intuition behind proof:

let $S = \{x_1, x_2, x_3, \ldots\}$. Then:

$\neg[\forall x \in S : P(x)]$

$\equiv \neg[P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots]$ infinite # of props — def. of $\forall$

$\equiv \neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \cdots$ de Morgan's Law

$\overset{\sim}{\equiv}$ $\exists x \in S : \neg P(x)$ def of $\exists$

<u>Thm</u> (3.42) $\neg[\exists x \in S : Q(x)] \Longleftrightarrow [\forall x \in S : \neg Q(x)]$

<u>Pf</u> let $P(x) = \neg Q(x)$.

$\neg[\forall x \in S : P(x)] \Longleftrightarrow [\exists x \in S : \neg P(x)]$ Thm 3.41

$\forall x \in S : P(x) \Longleftrightarrow \neg[\exists x \in S : \neg P(x)]$ negating both sides

$\forall x \in S : \neg Q(x) \Longleftrightarrow \neg[\exists x \in S : Q(x)]$ subs. $\square$

<u>ex</u> $\neg(\exists x \in \mathbb{R} : x^2 + 1 = 0) \equiv \forall x \in \mathbb{R} : x^2 + 1 \neq 0$

<u>Thm</u> (3.43) For $S \neq \emptyset$,

$$[\forall x \in S : P(x)] \Rightarrow [\exists x \in S : P(x)]$$

"if it's true for all, it's true for one"
"if everybody's doing it, then somebody's doing it"

<u>ex</u> $\forall x \in \mathbb{Z} : \text{isEven}(2x) \Rightarrow \exists x \in \mathbb{Z} : \text{isEven}(2x)$

<u>Pf</u> Suppose $\forall x \in S : P(x)$. WTS $\exists x \in S : P(x)$.

$\exists a \in S$ since $S \neq \emptyset$

$P(a)$ true $a \in S$ and given that $\forall x \in S : P(x)$

$\exists x \in S : P(x)$ it's a!

$\square$

Q what is the converse of 3.43?

For $S \neq \emptyset$, $[\exists x \in S : P(x)] \Rightarrow [\forall x \in S : P(x)]$.

Is it true? what does a disproof look like?

Thm $\forall x \in \emptyset : P(x)$ "$P(x)$ is vacuously true"

Pf Aiming for a contradiction, suppose the claim is false.

$\neg[\forall x \in \emptyset : P(x)]$ assumption

$\equiv \exists x \in \emptyset : \neg P(x)$ Thm 3.41

which is a contradiction, since there are no elements in $\emptyset$. $\square$

ex all even primes ≥ 30 are divisible by 10.

- This is true, bc there are no even primes ≥ 30.
- This is vacuously true.

Q: what is the negation (+ simplified) of:

The square of every real # is non-negative.

$$\neg[\forall x \in \mathbb{R} : x^2 \geq 0] \equiv \exists x \in \mathbb{R} : x^2 < 0$$

Q: Negate + simplify:

if a is odd then a^2 is odd.

$$\neg(\forall a \in \mathbb{Z} : a \text{ odd} \Rightarrow a^2 \text{ odd})$$

$$\equiv \exists a \in \mathbb{Z} : \neg(a \text{ odd} \Rightarrow a^2 \text{ odd})$$

$$\equiv \exists a \in \mathbb{Z} : (a \text{ odd and } a^2 \text{ not odd})$$

$$\hookrightarrow \neg(p \Rightarrow q)$$

$$\equiv \neg(\neg p \vee q)$$

$$\equiv p \wedge \neg q$$