Relations

CS application: relational databases.

tables hold relations

student id	first name	last name
123	Bob	smith
⋮	⋮	⋮

student id	passed course
123	CSCI 246
123	CSCI 112
⋮	⋮

Questions about data stored in tables can be posed precisely using the language of relations.

SQL (structured query language) implements this.

Def A binary relation R on sets A, B is a subset $R \subseteq A \times B$.

Recall: $A \times B = \{ \langle a, b \rangle : a \in A, b \in B \}$

we write $(x,y) \in R$ as $x R y$
$(x,y) \notin R$ as $x \not R y$

examples:

① "is related to" is a binary relation on people!

let P be the set of all people.

"is related to" is $\{\langle x, y\rangle : x, y \in P,\ x \text{ is related to } y\}$

So $\langle$Serena Williams, Venus Williams$\rangle$ $\in$ "is related to"

But $\langle$Lucy Williams, Serena Williams$\rangle$ $\notin$ "is related to"

② $<$ on $A = \{1, 2, 3, 4\}$ is

$< = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

So $1 < 2$ but $3 \not< 2$, because $\langle 1,2\rangle \in$ $<$ but $\langle 3,2\rangle$ is not.

③ let $f : A \to B$ be a function.

$\{\langle a, f(a)\rangle : a \in A\} \subseteq A \times B$, so it is a relation!

So any function defines a relation.

$f : A \to B$ is a function $\Rightarrow \{\langle a, f(a)\rangle : a \in A\}$ is a relation

<u>Q</u> Is the converse true?

$\{\langle x,y\rangle : x\in A, y\in B\}$ is a relation $\Rightarrow$ $f: A\to B$ s.t. $f(x)=y$ is a function

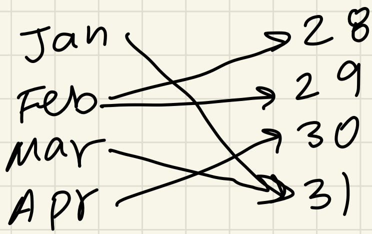
④ Let A = months, B = num. days

$\{\langle Jan, 31\rangle, \langle Feb, 28\rangle, \langle Feb, 29\rangle, \langle Mar, 31\rangle, \dots\}$

is the relation indicating the number of days in a month.

Note: we can also represent relations visually:

Jan	31
Feb	28
Feb	29
Mar	31
Apr	30
⋮	⋮



⑤ $A = \{1, 2, 3, 4\}$

$R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,4\rangle, \langle 2,2\rangle, \langle 4,2\rangle, \langle 4,4\rangle\}$

aRb is "has same parity as"

⑥ $R\cap < \;=\{\langle 1,3\rangle, \langle 2,4\rangle\}\subseteq A\times A$

so is a relation!

a has same parity as b and a is less than b.

Properties of relations

let $R \subseteq A \times A$ (so R is a relation on A)

let's represent R as a graph:

$$\begin{tikzcd} a_1 \arrow[r] \arrow[dr] & a_2 \arrow[loop right] \\ & a_3 \end{tikzcd}$$

$R = \{\langle a_1, a_2\rangle, \langle a_2, a_2\rangle, \langle a_1, a_3\rangle\}$

edge a_1 to a_2 means $\langle a_1, a_2\rangle$

R is <u>reflexive</u> if $\forall a \in A: \; aRa$

all nodes have self-loops

$$\begin{tikzcd} a_1 \arrow[loop right] \arrow[r, bend right] & a_2 \arrow[loop right] \end{tikzcd}$$

R is <u>irreflexive</u> if $\forall a \in A: \; a \not R a$

no node has a self-loop

$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[dl] \\ a_3 \arrow[u] & \end{tikzcd}$$

<u>Q</u> are all relations on A either reflexive or irreflexive?

R is <u>symmetric</u> if

$$\forall a_1, a_2 \in A: a_1 R a_2 \Rightarrow a_2 R a_1$$

$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[l, bend right] \arrow[r] & a_3 \arrow[l, bend right] \arrow[r] & a_4 \end{tikzcd}$$

whenever we have a forward edge, we also have a backward edge.

R is <u>anti-symmetric</u> if

$$\forall a_1, a_2 \in A : (aRb \land bRa) \Rightarrow (a = b)$$

$$\cancel{a_1 \rightleftarrows a_2} \qquad \begin{tikzcd} a_1 \arrow[loop left] \arrow[r] & a_2 \end{tikzcd}$$

never have backwards edge (but self-loops ok)

R is <u>transitive</u> if

$$\forall a, b, c \in A : (aRb \land bRc) \Rightarrow aRc$$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd} \qquad a \rightarrow \cancel{b} \rightarrow c$$

shortcut edges must always exist

<u>Q</u> is $a_1 \rightleftarrows a_2$ transitive?

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r, bend left] \arrow[r, bend right] & 3, 4 \end{tikzcd}$$

<u>ex</u>

aRb

relation $<$ on $\mathbb{Z}$. $1 < 2$ or $\langle 1, 2 \rangle \in R$

- reflexive : $\forall a \in \mathbb{Z} : aRa$ $\quad a < a$? no.

no. proof by counter example: $1 \not< 1$.

- irreflexive: $\forall a \in \mathbb{Z} : a \not R a$. yes!

proof: Let $a \in \mathbb{Z}$. $a \not< a$. $\square$

- symmetric : $\forall a, b \in \mathbb{Z} : a < b \Rightarrow b < a$.

no. proof by counter example: $1 < 2$ but $2 \not< 1$.

- anti-symmetric: $\forall a, b \in \mathbb{Z}: aRb \wedge bRa \Rightarrow a = b$

yes! proof: let $a, b \in \mathbb{Z}$.