Def A set is a collection of distinct, unordered items called elements.

ex $D = \{0,1,2,3,\dots,9\}$ has 10 elements
bits = $\{0,1\}$ has 2 elements
Bool = $\{\text{True}, \text{False}\}$ has 2 elements
$\mathbb{Z}$ = integers has ∞ elements
$\{\dots,-2,-1,0,1,2,\dots\}$
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = reals
$V = \{a,e,i,o,u,y\}$ has 6 elts.
$\Sigma = \{a,b,c,\dots,x,y,z\}$ has 26 elts.

Def Two set A, B are equal (denoted $A=B$) iff A and B contain exactly the same elts.

ex. $\{0,1\} = \{1,0\} = \{0,0,1\}$ (but we usually don't write down repeats)

Def We write $x \in S$ ($x \notin S$) iff x is in (not in) S.

ex. $0 \in$ bits $2 \notin$ bits $\pi \notin \mathbb{Z}$

Def The cardinality or size of a set S (denoted by $|S|$) is the number of distinct elements in S.

ex. $|\text{bits}| = 2$ $|\Sigma| = 26$

note: we don't consider infinity to be a number, so we don't write $|\mathbb{Z}|$ = anything. We just say "$\mathbb{Z}$ has infinite cardinality" or similar.

Q Can we have a set S such that (s.t.) $|S| = 0$?

Def The empty set, denoted $\{\}$ or $\emptyset$, is the set with no elements.

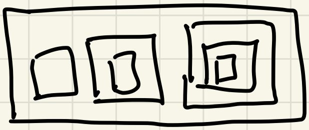
$|\emptyset| = 0$.

Note $\{\emptyset\} \neq \emptyset$ → empty box
↳ box containing an empty box

$|\{\emptyset\}| = 1$

$F = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ has 3 elements.

F is a box with 3 elements:



1. an empty box
2. a box containing an empty box
3. a box containing a box containing an empty box

Q If $A = B$ does $|A| = |B|$? yes, by substitution

Q Is the converse true? If $|A| = |B|$, does $A = B$?

Disproof by counter example:

Consider $A = \{5\}$, $B = \{c\}$.
$|A| = 1$ and $|B| = 1$. But $A \neq B$.

Def Set builder notation defines a set

$$S = \{x \underset{\text{"such that"}}{:} \text{a rule about } x\}$$

S contains the elements x s.t. the rule about x is true.

ex.

$\text{evens} = \{x : x \in \mathbb{Z} \text{ and } x \text{ even}\}$

$\text{evens} = \{x : x = 2c \text{ for } c \in \mathbb{Z}\}$

$\text{evens} = \{x \in \mathbb{Z} : x \text{ even}\}$

$\text{bits} = \{x \in \mathbb{Z} : 0 \leq x \leq 1\}$

Def A is a subset of B (denoted $A \subseteq B$) iff every element of A is also in B.
We can also say that B is a superset of A (denoted $B \supseteq A$).

ex.

$\text{evens} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

but $\mathbb{R} \not\subseteq \mathbb{Q} \not\subseteq \mathbb{Z} \not\subseteq \text{evens}$

$\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$

$1/2 \in \mathbb{Q}$ but $1/2 \notin \mathbb{Z}$

$3 \in \mathbb{Z}$ but $3 \notin$ evens

$\text{bits} \subseteq \{x : x \in \mathbb{Z} \text{ and } 0 \leq x \leq 9\}$

$\{0,5\} \subseteq \{0,1\}$

Note $\emptyset \subseteq S$ for any set S

$S \subseteq S$ for any set S

__Q__ If $A \subseteq B$ what can we say about $|A|, |B|$?

$|A| \leq |B|$, because every elt. of A also in B

__Q__ Is the converse true?

__claim__ If $|A| \leq |B|$, then $A \subseteq B$.

Disproof by counter example: Let $A = \{1\}$ and $B = \{2\}$. $|A| = 1$ and $|B| = 1$, so $|A| \leq |B|$. But $1 \in A$ and $1 \notin B$, so $A \not\subseteq B$. ☒

__claim__ $\{x \in \mathbb{Z} : 18 \overset{\text{divides}}{|} x\} \subseteq \{x \in \mathbb{Z} : 6 | x\}$

Step 1: understand notation, terms. Sometimes it's useful translate between math notation and English or vice versa.

- The set of numbers divisible by 18 is a subset of the numbers divisible by 6.
- Every number divisible by 18 is also div. by 6.

Step 2: do some examples.

ex.	x	$18 \mid x$?	$6 \mid x$?
	18	T	T
	6	F	T
	0	T	T
	1	F	F

what would a counter example look like? $\overset{18|x}{T}$ $\overset{6|x}{F}$

Pf WTS $\{x \in \mathbb{Z} : 18 | x\} \subseteq \{x \in \mathbb{Z} : 6 | x\}$
WTS $a \in \{x \in \mathbb{Z} : 18 | x\}$ then $a \in \{x \in \mathbb{Z} : 6 | x\}$
by def. of $\subseteq$

Suppose that $a \in \{x \in \mathbb{Z} : 18 | x\}$.

statement	reasoning
$a = 18c$ for $c \in \mathbb{Z}$	by def. of div. by 18
$a = 6 \cdot 3c$	by factoring
$a = 6 \cdot k$ for some $k \in \mathbb{Z}$	because product of ints is int $(3c)$
$6 \| a$	by def. of div. by 6
$a \in \{x \in \mathbb{Z} : 6 \| x\}$	rewriting ☒

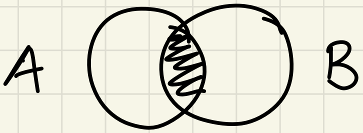
Def $A \cup B$ "A union B" is $\{x : x \in A \text{ or } x \in B\}$

note that elements $x \in A$ and $x \in B$ are in $A \cup B$.

ex. $\{2,4,6\} \cup \{2,3,4\} = \{2,3,4,6\}$
even ints $\cup$ odd ints $= \mathbb{Z}$
$\mathbb{R}^{\geq 0} \cup \mathbb{R}^{\leq 0} = \mathbb{R}$
reals greater or eq 0
reals l.t. or eq 0

$A \cup \emptyset = A$ for any set A
$A \cup A = A$ for any set A

Def $A \cap B$ "A intersect B" $\{x : x \in A \text{ and } x \in B\}$



ex. $\{2,4,6\} \cap \{2,3,4\} = \{2,4\}$ not disjoint

evens ∩ odds = ∅ disjoint

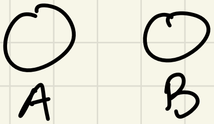
$A \cap \emptyset = \emptyset$ for all sets A disjoint

$A \cap A = A$ for all sets A not disjoint

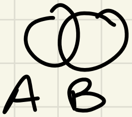
$\mathbb{R}^{\geq 0} \cap \mathbb{R}^{\leq 0} = \{0\}$ not disjoint

Def Sets A, B are disjoint if $A \cap B = \emptyset$.
That is, they have no elements in common.

i.e.

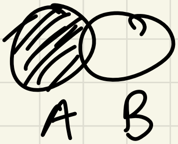


disjoint



not disjoint

Def $A - B$ or $A \setminus B$ "A minus B" $\{x \in A : x \in A \text{ and } x \notin B\}$



ex. $\{2,4,6\} - \{2,3,4\} = \{6\}$

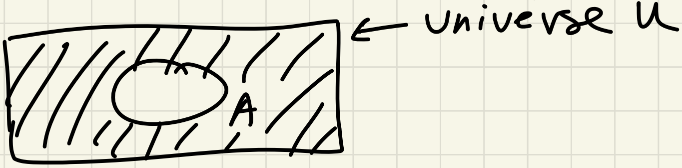
$\{2,3,4\} - \{2,4,6\} = \{3\}$

evens − odds = evens

$A - B \subseteq A$ for all sets A, B

$A - \emptyset = A$ for all sets A

Def $\bar{A}$ or $\sim A$ "A complement" $\{x : x \in A\}$



ex. $\overline{\{2,4,6\}} = \{0,1,3,5,7,8,9\}$ if U is $\{x \in \mathbb{Z} : 0 \leq x \leq 9\}$

$= \{\ldots -2,-1,0,1,3,5,7,8,9,\ldots\}$ if $U = \mathbb{Z}$

Claim $\overset{A}{\{x \in \mathbb{Z} : x \mid 2\}} \cap \overset{B}{\{x \in \mathbb{Z} : x \mid 9\}}$
$\subseteq \underset{C}{\{x \in \mathbb{Z} : 6 \mid x\}}$

Step 1: if x div. by 2 and x div. by 9, then x div. by 6.

Step 2: examples

x	$x \in A \cap B$?	$x \in C$?
6	F	T
0	T	T
3	F	T
18	T	T

what would be a counter example?

$x \in A \cap B$	$x \in C$
T	F

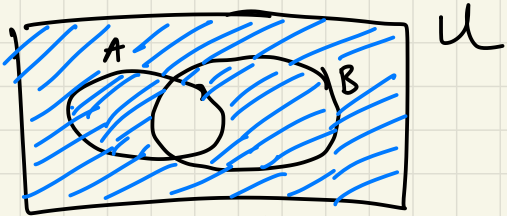
Pf Suppose $x \in A \cap B$. WTS $x \in C$.

statement	reasoning
$x \in A$ and $x \in B$	by def. of $\cap$
$2 \mid x$ and $9 \mid x$	by def. of A, B
$x = 2c$ and $x = 9d$ for $c, d \in \mathbb{Z}$	by def. of divisibility
$2c = 9d$	by substitution
$2 \mid 9d$	by def. of $\mid$ (we wrote $9d$ as $2c$)
d is even	$2 \mid 9d$, but 9 is odd, so d must be even
$d = 2d'$ for $d \in \mathbb{Z}$	by def. of even
$x = 9(2d')$	by substitution
$x = 3 \cdot 3 \cdot 2 \cdot d'$	by factoring
$x = 6 \cdot 3 \cdot d'$	by mult.
$6 \mid x$	because $3d' \in \mathbb{Z}$
$x \in C$	by def. of C ☒

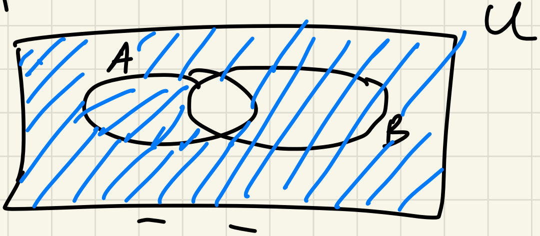
goal: $x = 6k$ $k \in \mathbb{Z}$

Thm (De Morgan's) $\overline{A \cap B} = (\bar{A} \cup \bar{B})$

The complement of the intersection is the union of the complements.



$\overline{A \cap B}$



$\bar{A} \cup \bar{B}$

Pf We will prove that

$$\left.\begin{array}{l} \overline{A \cap B} \subseteq (\bar{A} \cup \bar{B}) \text{ and} \\ (\bar{A} \cup \bar{B}) \subseteq \overline{A \cap B} \end{array}\right\} \Rightarrow \begin{array}{l} \overline{A \cap B} = (\bar{A} \cup \bar{B}) \\ x^{\in} \text{ iff } x^{\in} \end{array}$$

$\overline{A \cap B} \subseteq (\bar{A} \cup \bar{B})$: we will show that if $x \in \overline{A \cap B}$, then $x \in (\bar{A} \cup \bar{B})$.

Suppose $x \in \overline{A \cap B}$. WTS $x \in \bar{A} \cup \bar{B}$.

$x \notin A \cap B$	by def. of $\overline{\quad}$
x not in both A and B	by def. of $\notin$ and $\cap$
$x \notin A$ or $x \notin B$	by reasoning formalized later
$x \in \bar{A}$ or $x \in \bar{B}$	by def. of $\overline{\quad}$
$x \in \bar{A} \cup \bar{B}$	by def. of $\cup$

Now suppose $x \in \bar{A} \cup \bar{B}$. WTS $x \in \overline{A \cap B}$. $x \in \bar{A}$ or $x \in \bar{B}$ by definition of $\cup$. So we will prove by cases.

Case 1 : $x \in \bar{A}$. WTS $x \in \overline{A \cap B}$.

$x \notin A$	by def. of A
$x \notin A \cap B$	since $A \cap B \subseteq A$
$x \in \overline{A \cap B}$	by def. of $\overline{\quad}$

Case 2 : $x \in \bar{B}$. WTS $x \in \overline{A \cap B}$.

symmetric - replace A w/ B and viceversa.

Since the cases are exhaustive, the claim is proved. ☒

<u>Def</u> Given a set S, the <u>power set</u> of S (denoted $P(S)$) is the set of all subsets of S.

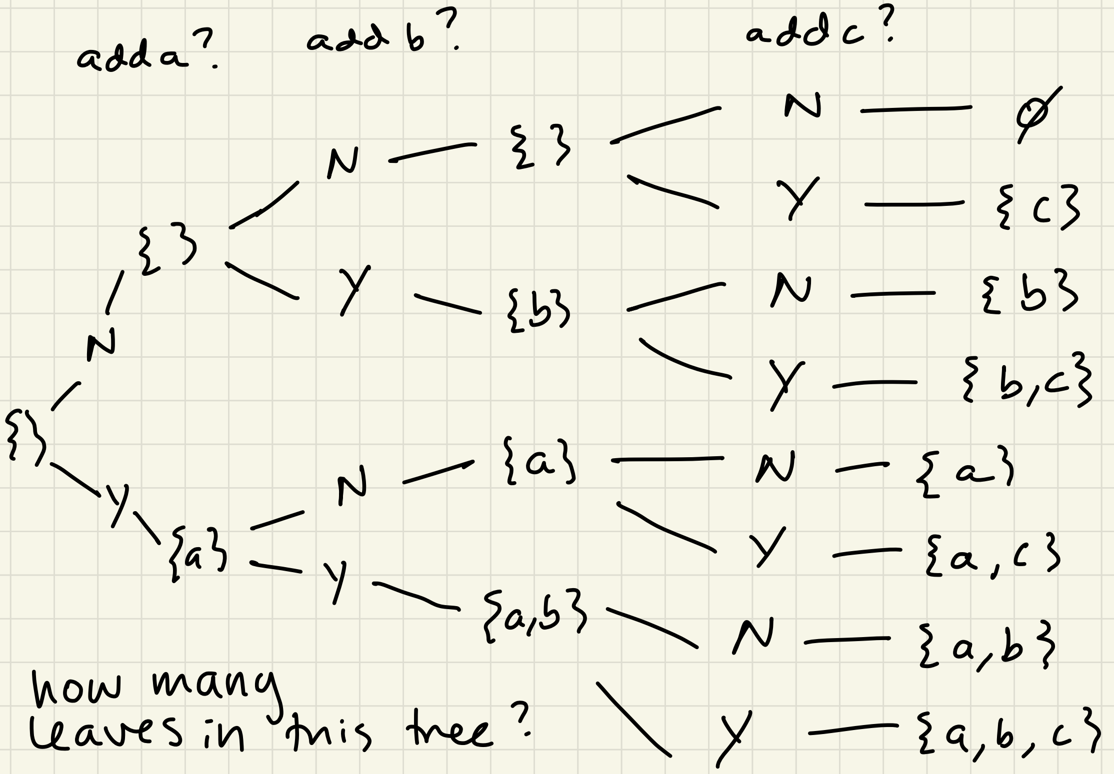
$P(S) = \{A : A \subseteq S\}$.

ex. $S = \{1, 2, 3\}$.

$P(S)$
"

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Another way to think about this: for every element of S, we either add it or don't!



how many leaves in this tree?
8

fact $|\mathcal{P}(S)| = 2^{|S|}$

ex. $B = \{1, 2, \{1,3\}\}$. $|B| = 3$. $|\mathcal{P}(B)| = 8$.

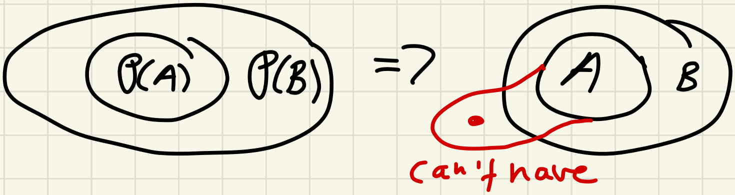
$\{\emptyset, \{1\}, \{2\}, \{\{1,3\}\}, \{1,2\}, \{1, \{1,3\}\}, \{2, \{1,3\}\},$
$\{1, 2, \{1,3\}\}\}$

Note Power set is also denoted 2^S for set S.

$\emptyset \in 2^S$ for all sets S

$S \in 2^S$ for all sets S

claim if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.



ex B from before – $B = \{1, 2, \{1,3\}\}$.
$A = \{1\}$. $\mathcal{P}(A) = \{\emptyset, \{1\}\}$.

$\mathcal{P}(A) \subseteq \mathcal{P}(B)$ $\quad$ $A \subseteq B$
T $\quad$ T

pf (direct)

Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. WTS $A \subseteq B$.

Suppose if $C \in \mathcal{P}(A)$ then $C \in \mathcal{P}(B)$, then if $y \in A$ then $y \in B$.

So we WTS if $y \in A$ then $y \in B$, and we have if $C \in \mathcal{P}(A)$ then $C \in \mathcal{P}(B)$ to work with.

Suppose $y \in A$.

$\{y\} \subseteq A$	by def. of $\subseteq$
$\{y\} \subseteq \mathcal{P}(A)$	by def. of $\mathcal{P}(A)$
$\{y\} \subseteq \mathcal{P}(B)$	by $\mathcal{P}(A) \subseteq \mathcal{P}(B)$
$y \in B$	by def. of $\mathcal{P}(B)$
$A \subseteq B$	by def. of $\subseteq$ $\boxtimes$

Def A sequence / list / tuple / array is an ordered collection of objects.

ex. $\langle 0,1 \rangle$ ⤵ these are not the same

$\langle 1,0 \rangle$

$\langle 0,0 \rangle$

$A = \langle a_1, a_2, a_3, a_4, \dots, a_n \rangle$ array of n elements

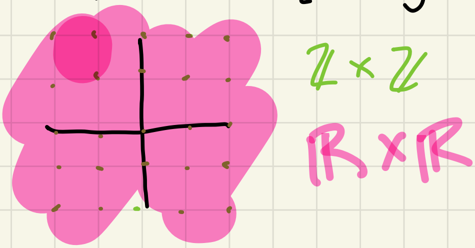
Def Let A, B be sets. The cartesian product $A \times B$ is the set of ordered pairs drawn from A and B in that order.

so $A \times B = \{ \langle a,b \rangle : a \in A \text{ and } b \in B \}$

ex. $\{a,b,c\} \times \{0,1\} = \{ \langle a,0 \rangle, \langle b,0 \rangle, \langle c,0 \rangle, \langle a,1 \rangle, \langle b,1 \rangle, \langle c,1 \rangle \}$

$\mathbb{Z} \times \mathbb{Z} = \{ \langle x,y \rangle : x \in \mathbb{Z}, y \in \mathbb{Z} \}$
all integer points in 2D plane

$\mathbb{R} \times \mathbb{R} = \{ \langle x,y \rangle : x \in \mathbb{R}, y \in \mathbb{R} \}$ all points in 2D plane



Q what is $|A \times B|$? $|A| \times |B|$.

Def For set S, S^n is $\overbrace{S \times S \times \cdots \times S}^{n \text{ times}}$

so $\{\langle s_1, s_2, \ldots, s_n \rangle : s_i \in S\}$

ex. $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
all length three bits

$\mathbb{R}^n$ = n-dimensional space

claim $A \times (B \cup C) = (A \times B) \cup (A \times C)$

step 1: notation, terms... ✓

this looks like the distributive property from regular arithmetic: $a(b+c) = ab + bc$.

step 2: examples.

$A = \{1, 2\}$ $B = \{b\}$ $C = \{\square, 0\}$

$A \times (B \cup C) = A \times \{b, \square, 0\} = \{1b, 1\square, 10, 2b, 2\square, 20\}$
$A \times B = \{1b, 2b\}$
$A \times C = \{1\square, 10, 2\square, 20\}$
$(A \times B) \cup (A \times C) = \{1b, 2b, 1\square, 10, 2\square, 20\}$

Pf We will prove $\subseteq$ and $\supseteq$ separately.

$\subseteq$: Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
That is, if $y \in A \times (B \cup C)$, then $y \in (A \times B) \cup (A \times C)$.

Suppose $y \in A \times (B \cup C)$.

$y \in \langle a, d \rangle$ where $a \in A$ and $d \in (B \cup C)$ by def. of $\times$

There are two cases: either $d \in B$ or $d \in C$.

Case 1: $d \in B$.

$y = \langle a, d \rangle \in A \times B$ by def. of X

$y \in (A \times B) \cup (A \times C)$ $\cup$ only adds elts to $A \times B$

case 2: $d \in C$.
symmetric.

$\subseteq$ is done.

$\supseteq$: Prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.
WTS if $y \in (A \times B) \cup (A \times C)$ then $y \in A \times (B \cup C)$.

Suppose $y \in (A \times B) \cup (A \times C)$.

case 1: $y \in A \times B$. exercise: show $y \in A \times (B \cup C)$
case 2: $y \in A \times C$. show $y \in A \times (B \cup C)$

Some more set-related notation.

ex. $S = \{2, 4, 3\}$

$\min\limits_{x \in S} x$ minimum elt. in S $\min\limits_{x \in S} x = 2$

$\max\limits_{x \in S} x$ max elt. in S $\max\limits_{x \in S} x = 4$

$\sum\limits_{x \in S} x$ sum of elts in S $\sum\limits_{x \in S} x = 9$

$\prod_{x \in S} x$ product of elts of S $\prod_{x \in S} x = 2 \cdot 4 \cdot 3$
$= 24$

$\prod_{x \in S} x^2 = 2^2 \cdot 4^2 \cdot 3^2$
$= 4 \cdot 16 \cdot 9$

If we have sets $A_1, A_2, \dots, A_n$, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

<u>Def</u> A <u>partition</u> is a set of subsets of S s.t.
- every elt of S is in some subset
- no elt of S is in more than one subset

ex. $\{2, 3, 4\}$

$\{\{2,3\}, \{4\}\}$ ✓

$\{\{2,3\}\}$ ✗