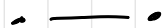
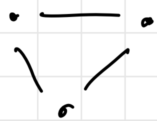
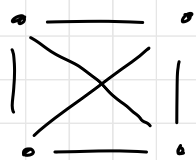
(pronounced "kleek")

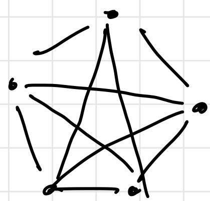
Def A <u>complete graph</u> or <u>clique</u> is an undirected graph $G = (V, E)$ s.t.

$$\forall u, v \in V \quad u \neq v \Rightarrow \{u, v\} \in E$$

The clique on n nodes is denoted K_n.

ex K_1  0 K_2  1

K_3  3 K_4  6

K_5  10

Q what is the relationship between $n = |V|$ and $m = |E|$ for K_n?

Claim K_n has $\dfrac{n(n-1)}{2}$ edges. $(n \geq 1)$

Proof #1 We give a way to count the edges and show that it gives $\dfrac{n(n-1)}{2}$.

Label the nodes $v_1, v_2, \dots, v_n$. Starting w/ v_1, count the uncounted edges and add to the total.

v_1 has $n-1$ uncounted edges.

v_2 has $n-2$ uncounted edges.

$\vdots$

v_{n-1} has 1 uncounted edges

v_n has 0 uncounted edges.

so $|E| = (n-1) + (n-2) + \cdots 1 = \frac{(n-1)n}{2}$

because $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

<u>proof #2</u> In K_n, every node has degree $n-1$. So

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (n-1) = n(n-1).$$

But by the handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$n(n-1) = 2|E|$$

$$\frac{n(n-1)}{2} = |E|$$

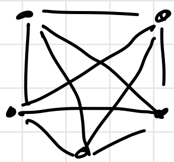
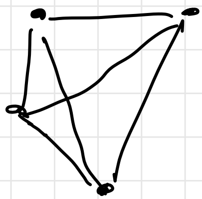
<u>Proof #3</u> Let $P(n)$ denote that K_n has $\frac{n(n-1)}{2}$ edges. We prove $\forall n \geq 1 : P(n)$ using mathematical induction on n.

Base case: $n = 1$. $\bullet = K_1$ has 0 edges.

$\frac{1(1-1)}{2} = 0$ so $P(1)$ holds.

Inductive case: we WTS $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$.
Assume $P(n-1)$. That is, assume K_{n-1} has $\frac{(n-1)(n-2)}{2}$ edges.

Now, consider an arbitrary clique K_n. Let K_n' be the graph created by removing one node and all of its incident edges.

(example: K_5:  K_5': )

Note that $K_n' = K_{n-1}$.

edges of K_n = # edges of K_{n-1} + # edges we have to add back to K_{n-1} to get K_n

$$= \frac{(n-1)(n-2)}{2} + n-1$$

$$= \frac{n^2-3n+2}{2} + \frac{2(n-1)}{2}$$

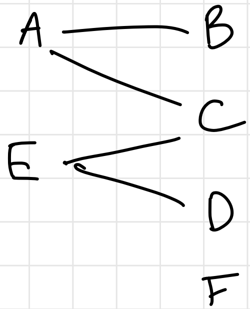
$$= \frac{n^2-3n+2+2n-2}{2}$$

$$= \frac{n^2-n}{2}$$

$$= \frac{n(n-1)}{2}$$

Def A <u>bipartite graph</u> $G = (L \cup R, E)$ s.t. $L \cap R = \emptyset$ (L, R disjoint) and $E \subseteq \{\{\ell, r\} : \ell \in L, r \in R\}$

ex



$L = \{A, E\}$
$R = \{B, C, D\}$

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$R = \{B, C, D, F\}$

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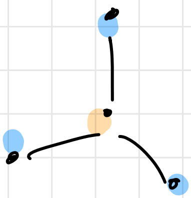

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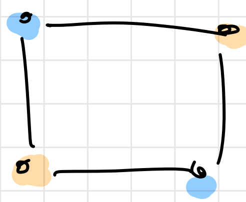

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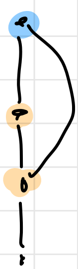
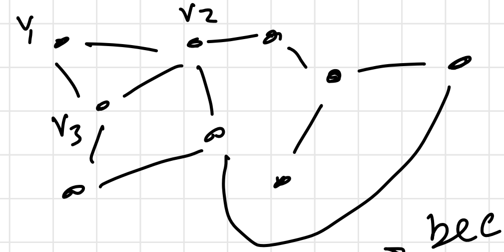


✗

claim If G contains a Δ (K_3), then it is not bipartite.

Proof Aiming for a contradiction, suppose that G contains a Δ and it is bipartite.

Let v_1, v_2, v_3 be the nodes of the Δ.

(example: 

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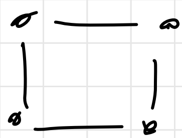
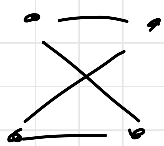
because we could reorder them

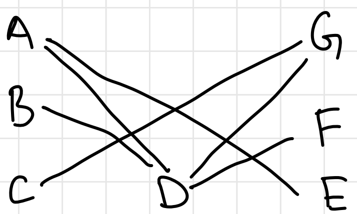
without loss of generality, suppose $v_1 \in L$. Then $v_2 \in R$. Since $v_2 \in R$, $v_3 \in L$. But there is an edge from v_1 to v_3 and both are in R, which contradicts that G is bipartite.

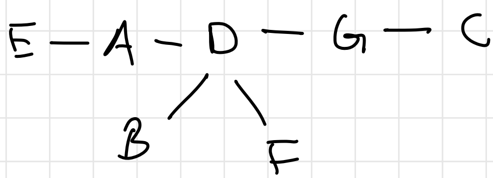
Note: two graphs are equal if they have the same nodes and edges.

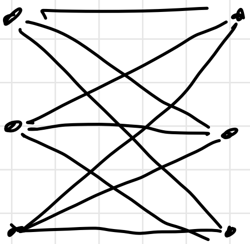
<u>Def</u> A graph is <u>planar</u> if we can draw it in the plane without the edges crossing.

<u>ex</u>

 = 



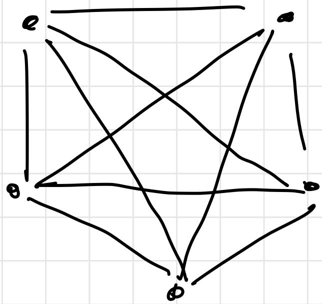




K33

×

K5



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