Last time

An algorithm is efficient if it does qualitatively better than brute force on every input.

↓ 2^n or worse, an^b (polynomial) = efficient

Poll:
Does every computational problem have a polynomial time algorithm?

1. yes
2. no

432

P = NP

P ≠ NP

Are we satisfied? No

How to define runtime:

① level of detail

→ big O notation

② which inputs?

best case ✗ → one bad input = unusable

average case → probability distribution over inputs

worst case → "for all inputs"

How many primitive operations does the algorithm take?

Algorithm 1

```
Input: integer array A of length n
for i = 1, 2, …, n :
    total = 0
    for j = 1, 2, …, n:
        total = total + A[i]
    B[i] = total
```

- for i = 1, 2, …, n : assign i — 1 prim op
- total = 0 : assign total — 1 p.o.
- inner loop:
  - for j = 1, 2, …, n: — assign j — 1 prim op
  - total = total + A[i] — retrive total, A[i] (2), + (1), assign total (1)
  - 5 prim ops n times
- B[i] = total : retrive total, assign B[i] — 2 p.o.

2 + 5n + 2 per outer loop

Assume the following

- *1 operation for basic arithmetic operations*
- *1 operation for variable assignment*
- *1 operation for variable retrieval*

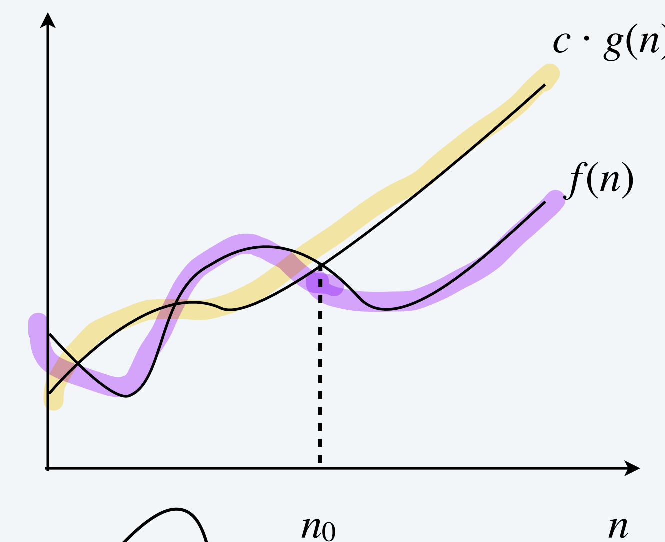
runtime = # primitive ops for input of size n

$$T(n) = n(5n+4) = 5n^2 + 4n$$

Big O notation

upper bound

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



ex $5n^2+4n$ is $O(n^2)$.

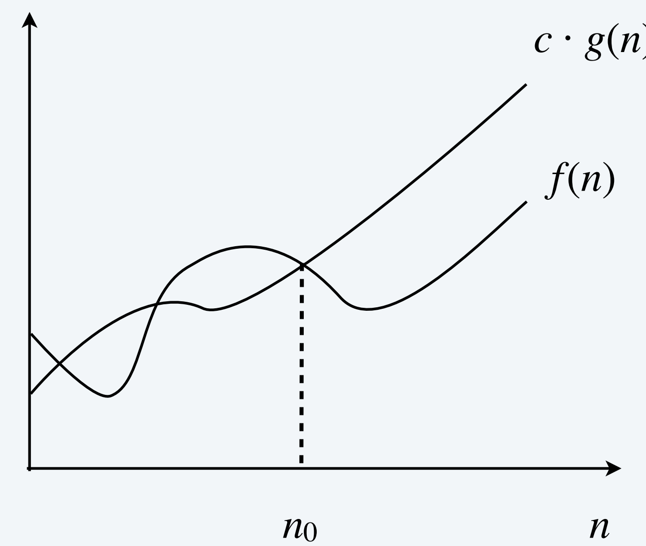
We need to show that there exist c and n_0 s.t.
$5n^2+4 \leq cn^2$ for all $n \geq n_0$.

how about $c=10$ and $n_0=0$?
Show that $5n^2+4n \leq 10n^2$ for $n \geq 0$.
Note that for all $n \geq 0$, $n^2 \geq n$.
So $5n^2+4n \leq 5n^2+4n^2 = 9n^2 \leq 10n^2$
could I have chosen $c=4$, $n_0=0$?
is $5n^2+4n \leq 4n^2$ for all $n \geq 0$?
could I have chosen $c=9$, $n_0=100$?

$5n^2+4n \leq 9n^2$ for all $n \geq 100$.

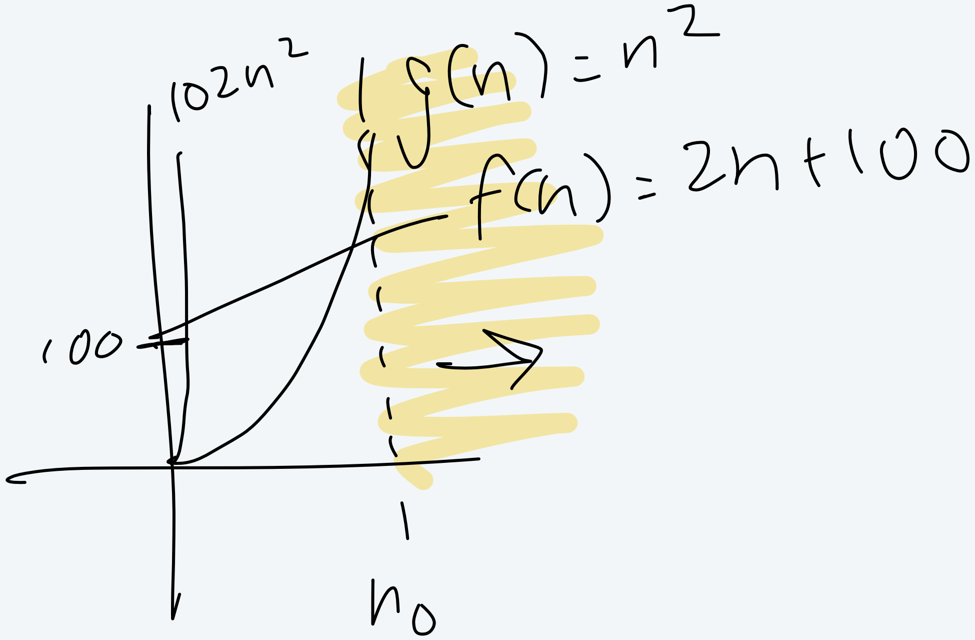
Big O notation: another example

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



Claim: $2n + 100$ is $O(n^2)$.

To show $2n+100$ is $O(n^2)$, we have to show that there exist constants $c>0$ and $n_0 \geq 0$ s.t. $2n+100 \leq cn^2$ for all $n \geq n_0$.



try $c = 102$
$n_0 = 1$

need to show:
$2n+100 \leq 102n^2$ for all $n \geq 1$

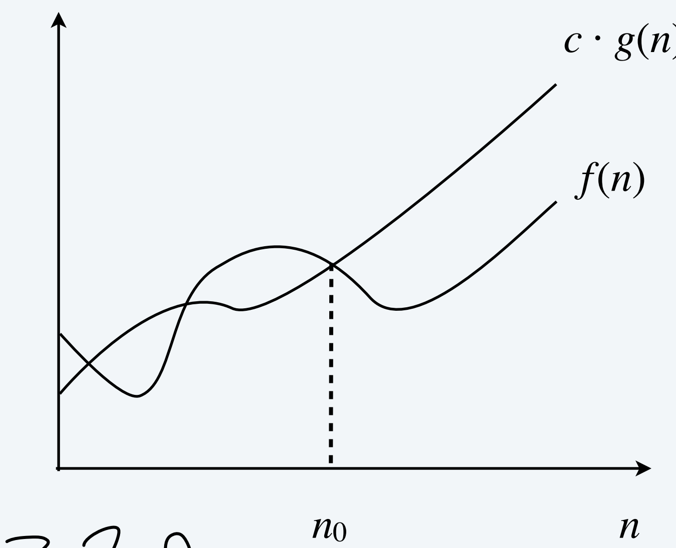
when $n=1$:
$2+100 \leq 102 \cdot 1^2$ ✓

$c=1$?
140 400
need $2n+100 \leq n^2$ for all $n \geq 20$

Big O notation: another example

$\leq$

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

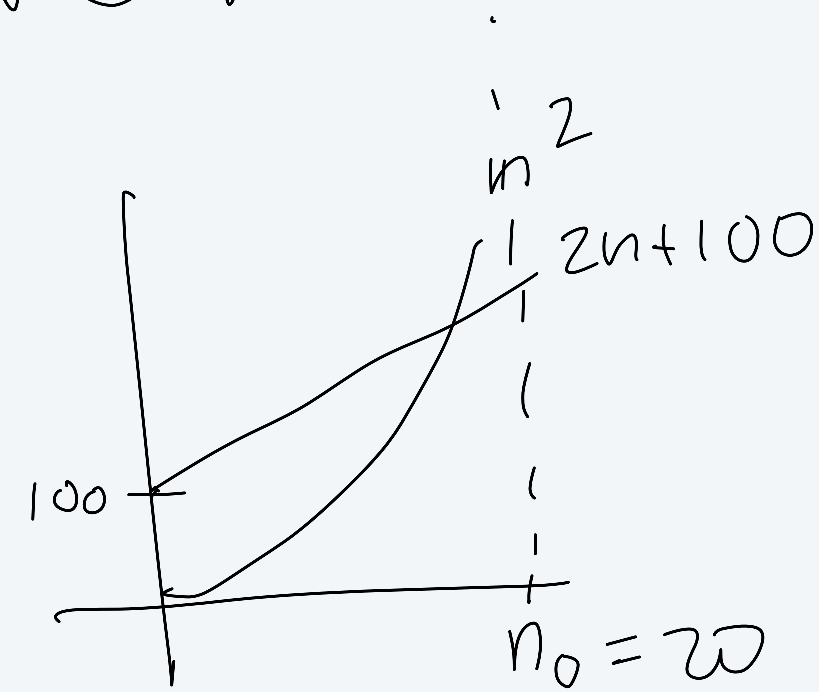


Claim: $2n + 100$ is $O(n^2)$.

Proof:

choose $c = 1$ and $n_0 = 20$. Then, we have $2n + 100 \leq n^2$ for all $n \geq 20$.

See picture.



Claim: 10 is $O(2^n)$.

is $5n+4$ $O(n)$?

how do I prove that

$5n^2+4$ is not $O(n)$?

To prove true: give $c>0$, $n_0 \geq 0$ s.t.

$5n^2+4 \leq cn$ for all $n \geq n_0$.

To prove false ($5n^2+4$ is not $O(n)$):

show that there are no $c>0$, $n_0 \geq 0$

s.t $5n^2+4 \leq cn$ for all $n \geq n_0$.

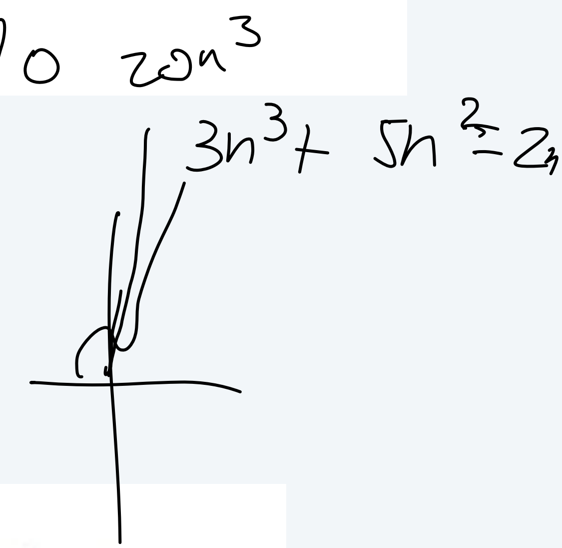
Let's do some examples together

1. Prove that $3n^3 + 5n^2 - 2n = O(n^3)$ using the definition of big O. That is, you must give a c and an n_0 and show that they fit the definition, either by reasoning in words or by drawing a graph.

$3n^3 + 5n^2 - 2n \leq cn^3$ for all $n \geq n_0$

$c = 20$, $n_0 = 0$.

$3n^3 + 5n^2 - 2n \leq 20n^3$ for all $n \geq 0$.



2. How would you prove that $3n^3 + 5n^2 - 2n \neq O(n^2)$? Don't do it, just tell me how you would.

Def of big O: $f(n)$ is $O(g(n))$ if there exist $c > 0$, $n_0 \geq 0$ s.t. $f(n) \leq c\,g(n)$ for all $n \geq n_0$.

1. (8 points) Suppose you have algorithms with the following runtimes. (Assume these are exact running times as a function of the input size n, not a asymptotic running times.) How much slower do these algorithms get when you double the input size?

   (a) n

   (b) $n \log n$

2. (7 points) Recall the definition of big O:

$f(n)$ is $O(g(n))$ if there exist positive constants n_0, c such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Using this definition, prove that $2n + 5$ is $O(n^2)$.