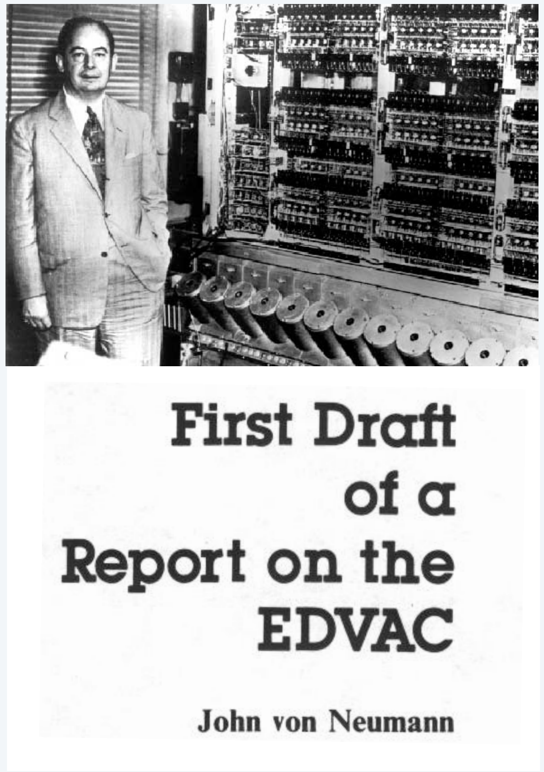
Today:

- merge sort → analyzing runtime of recursive algs
- proving a lower bound on sorting runtime
- counting inversions
- survey

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.



input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

n/2 log n

sort right half

A	G	L	O	R	H	I	M	S	T

merge results

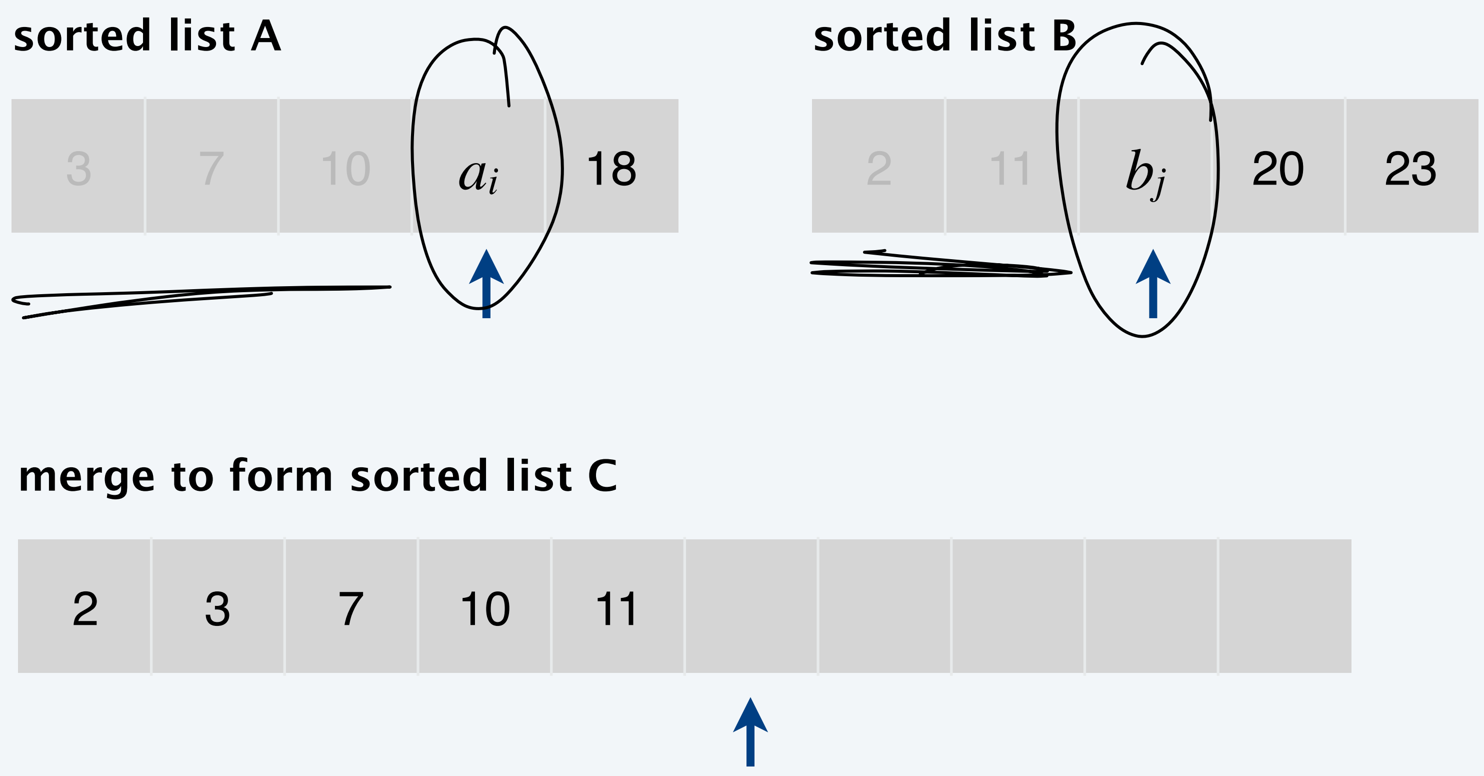
A	G	H	I	L	M	O	R	S	T

n

Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \le b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).



Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

Merge demo

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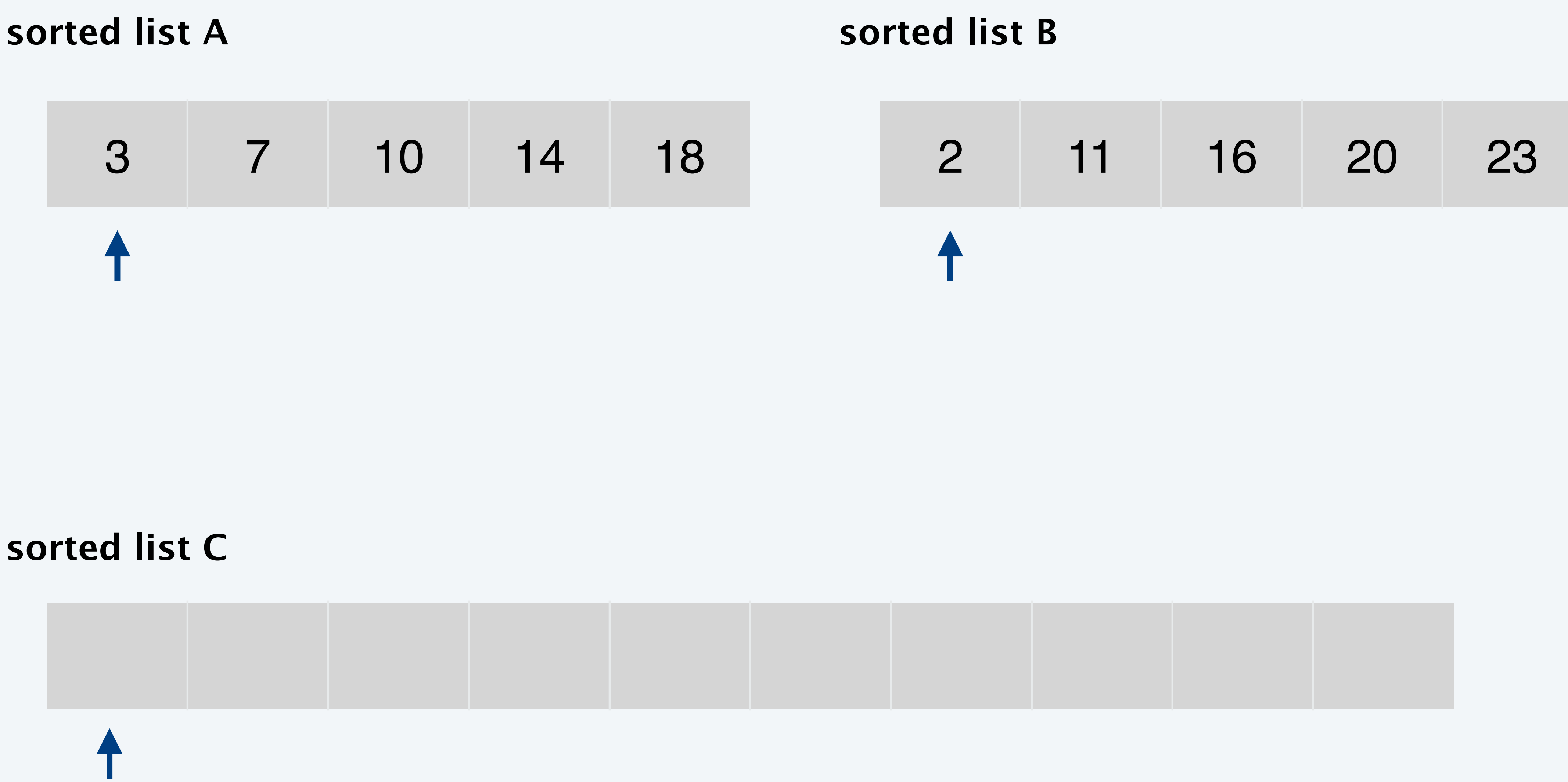


Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

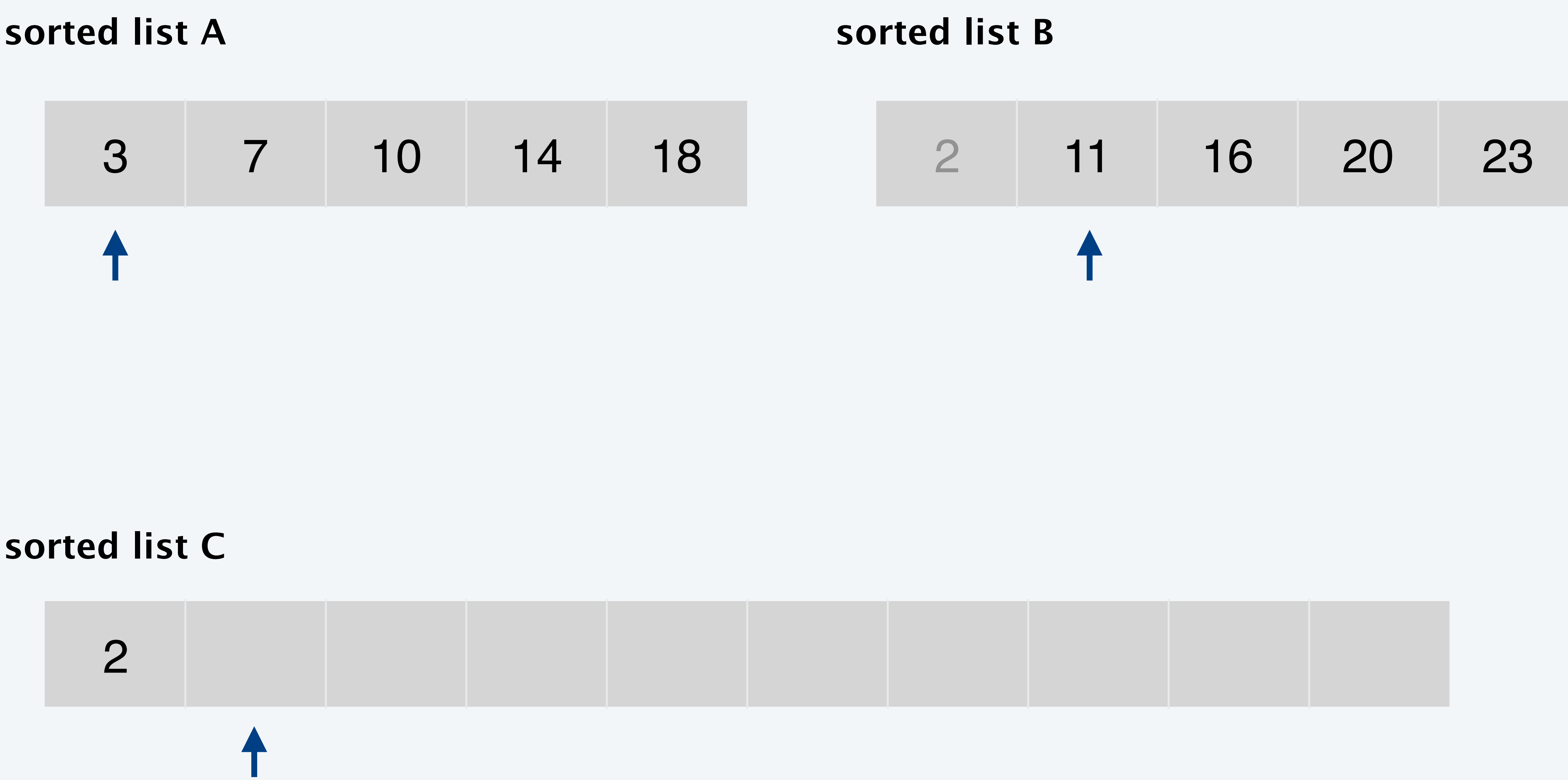
2	11	16	20	23

compare minimum entry in each list: copy 2

sorted list C

Merge demo

Given two sorted lists A and B, merge into sorted list C.



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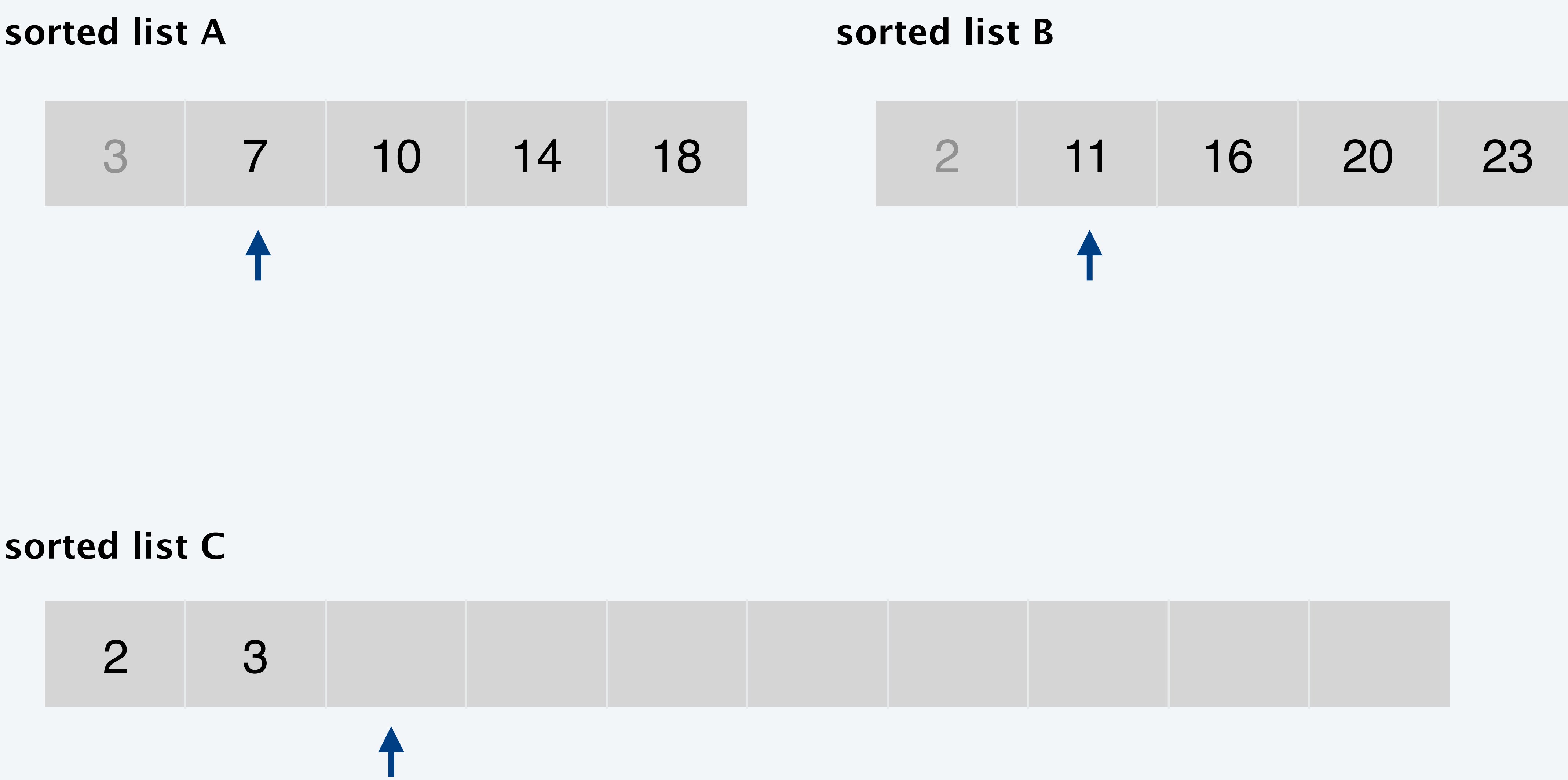


Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

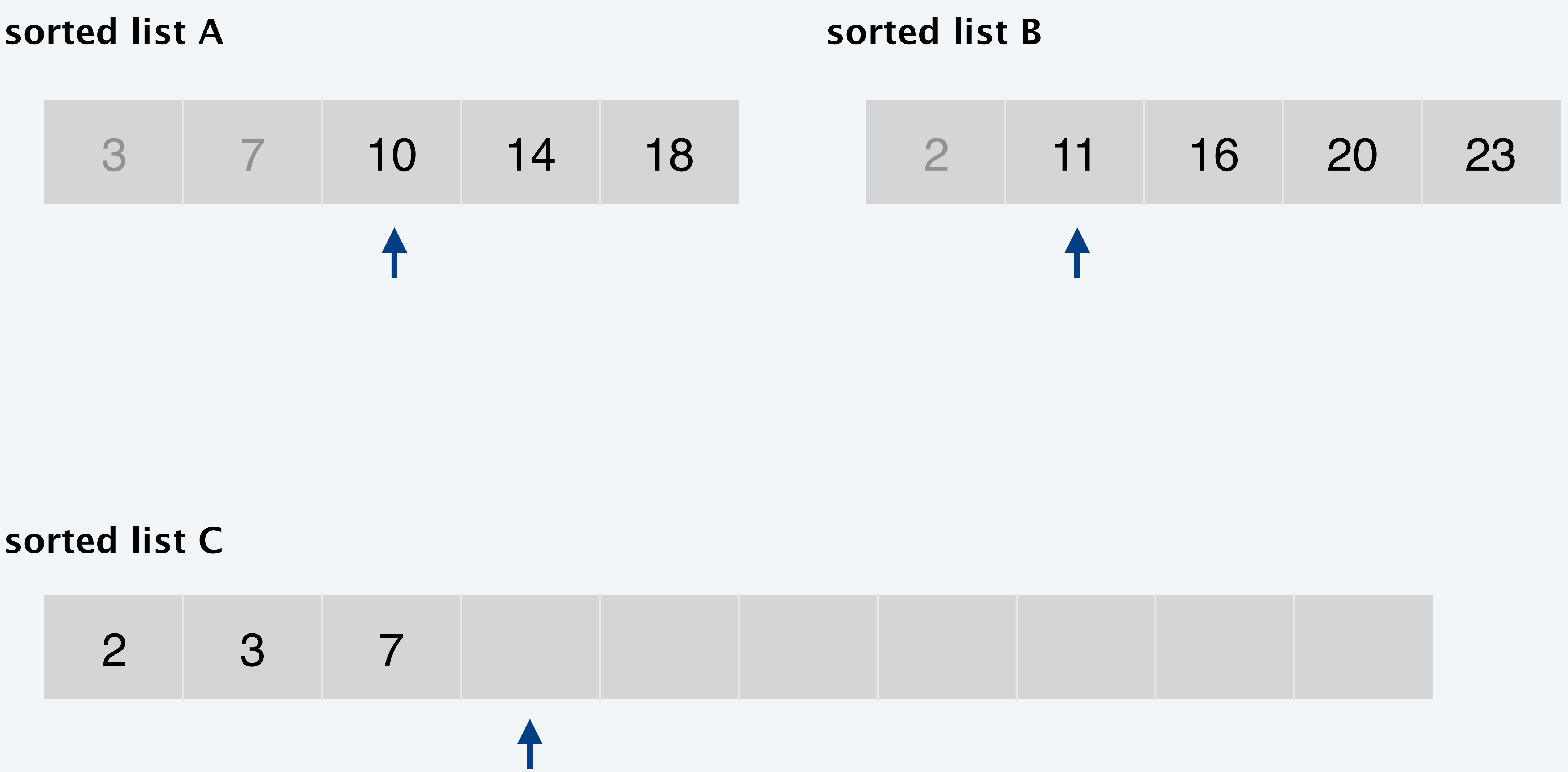
compare minimum entry in each list: copy 7

sorted list C

2	3								

Merge demo

Given two sorted lists A and B, merge into sorted list C.



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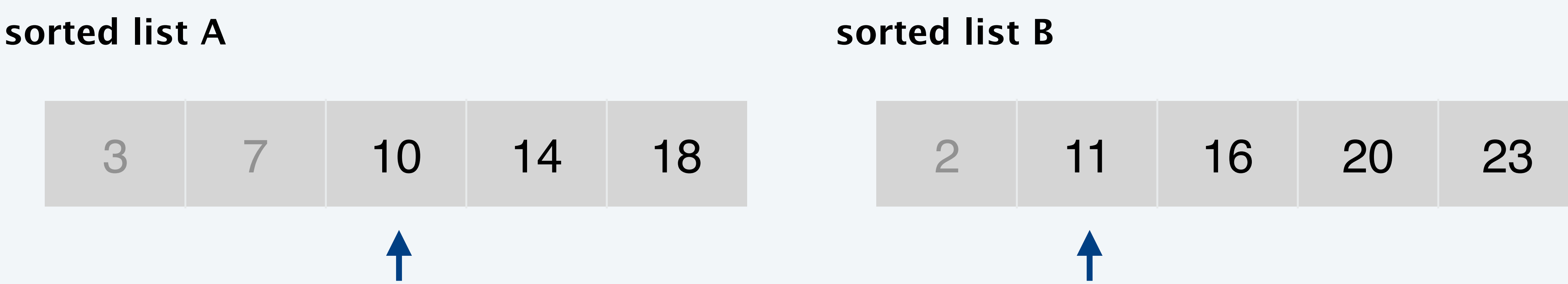
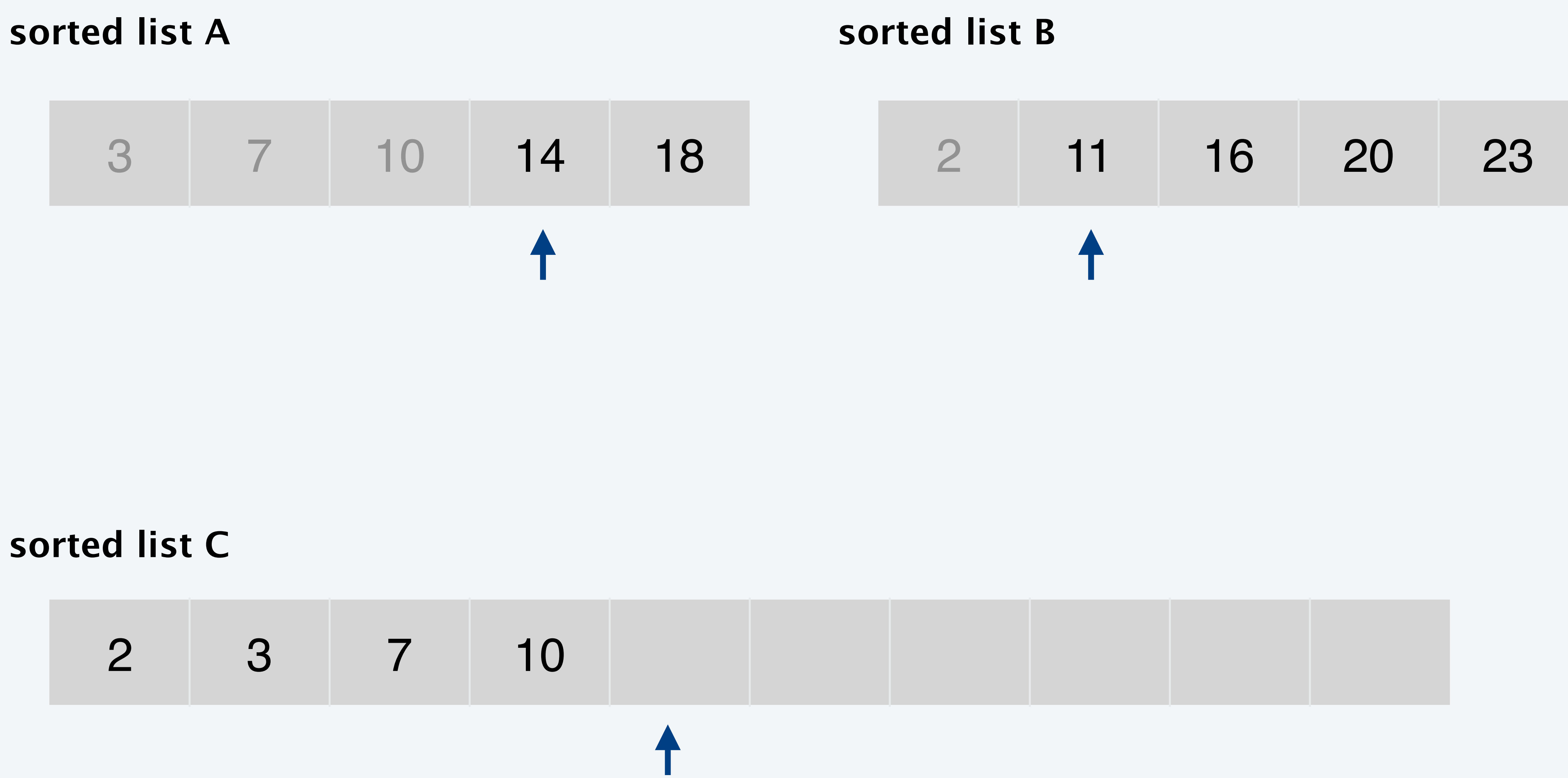


compare minimum entry in each list: copy 10

Merge demo

Given two sorted lists A and B, merge into sorted list C.



Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

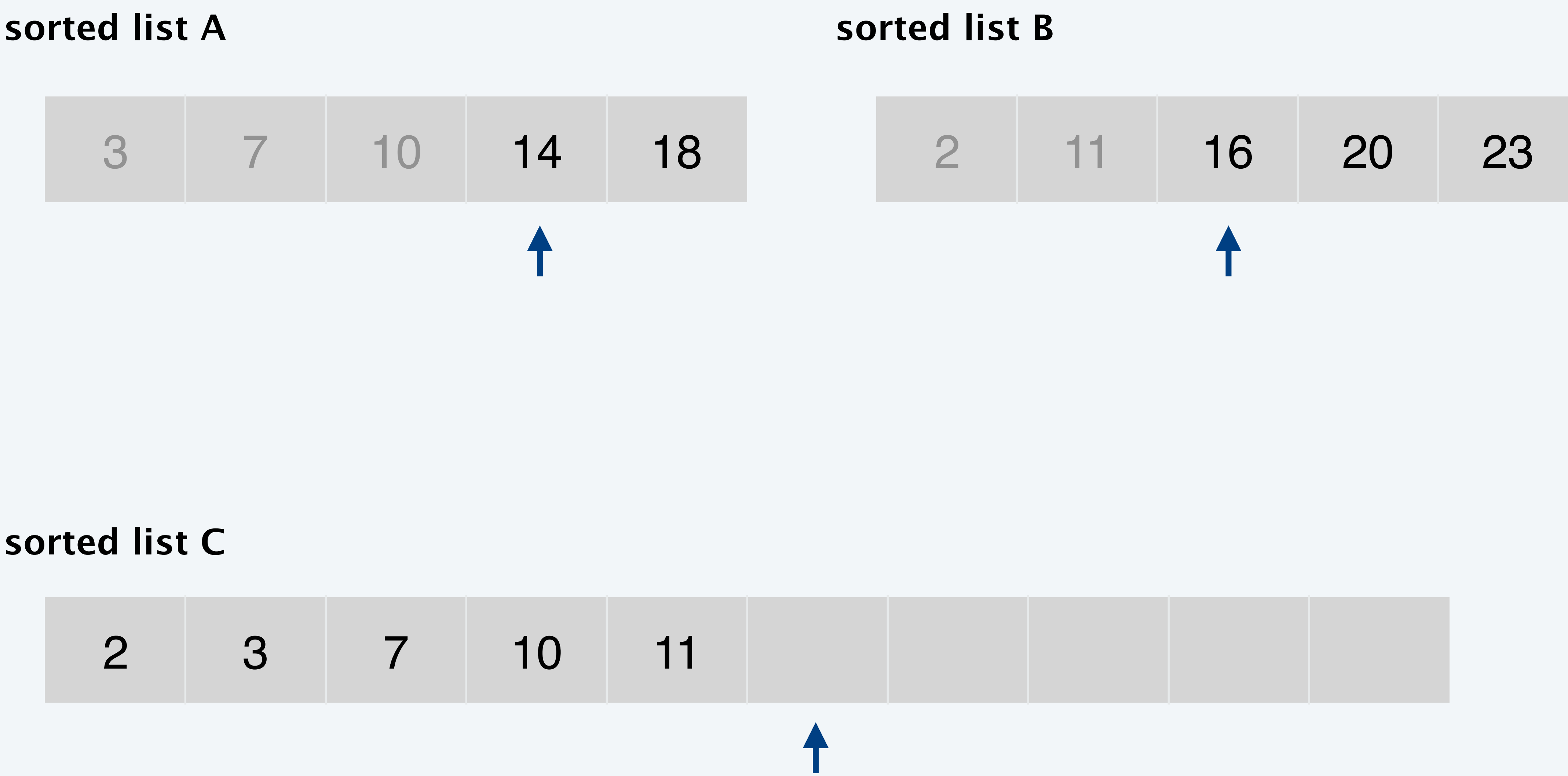
compare minimum entry in each list: copy 11

sorted list C

2	3	7	10						

Merge demo

Given two sorted lists A and B, merge into sorted list C.



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2	11	16	20	23

sorted list C

2	3	7	10	11	14				

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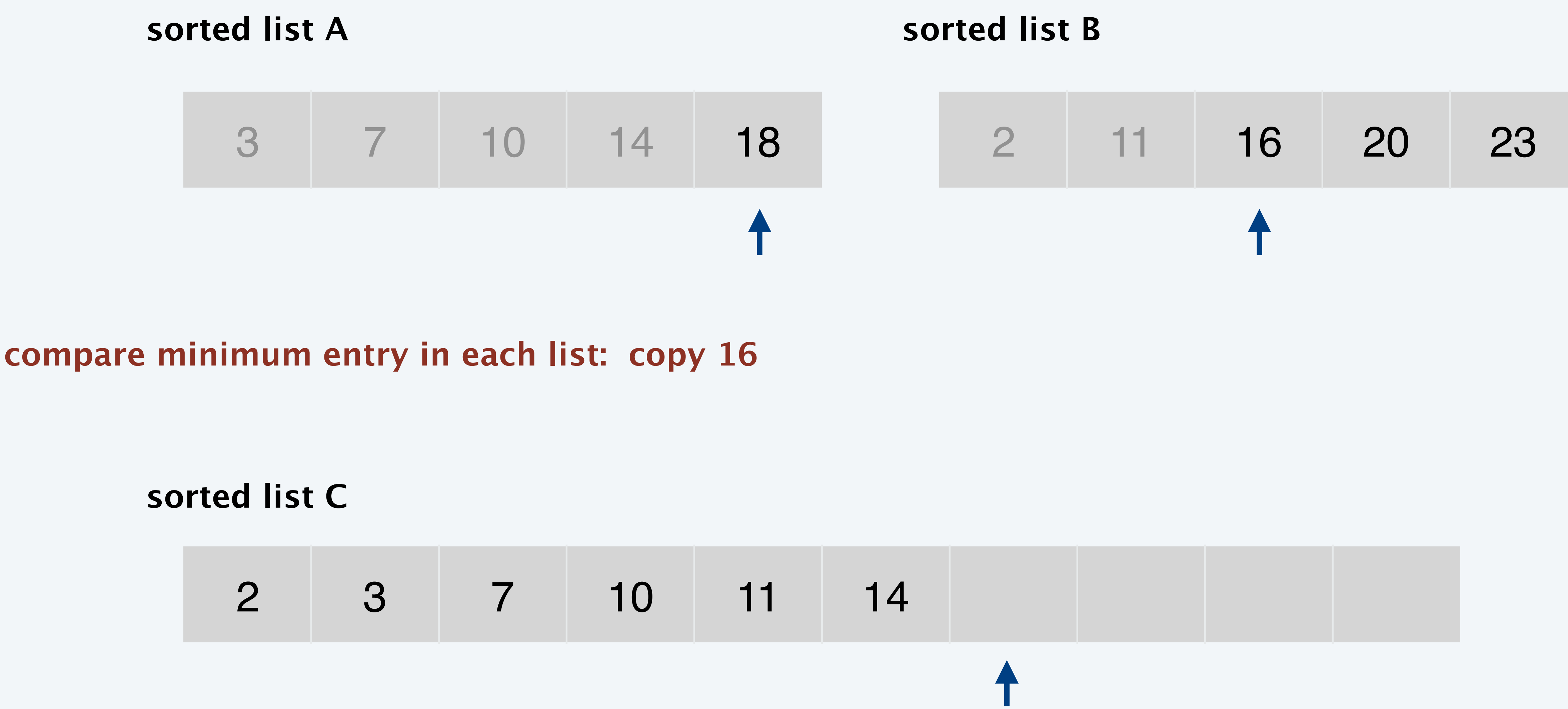

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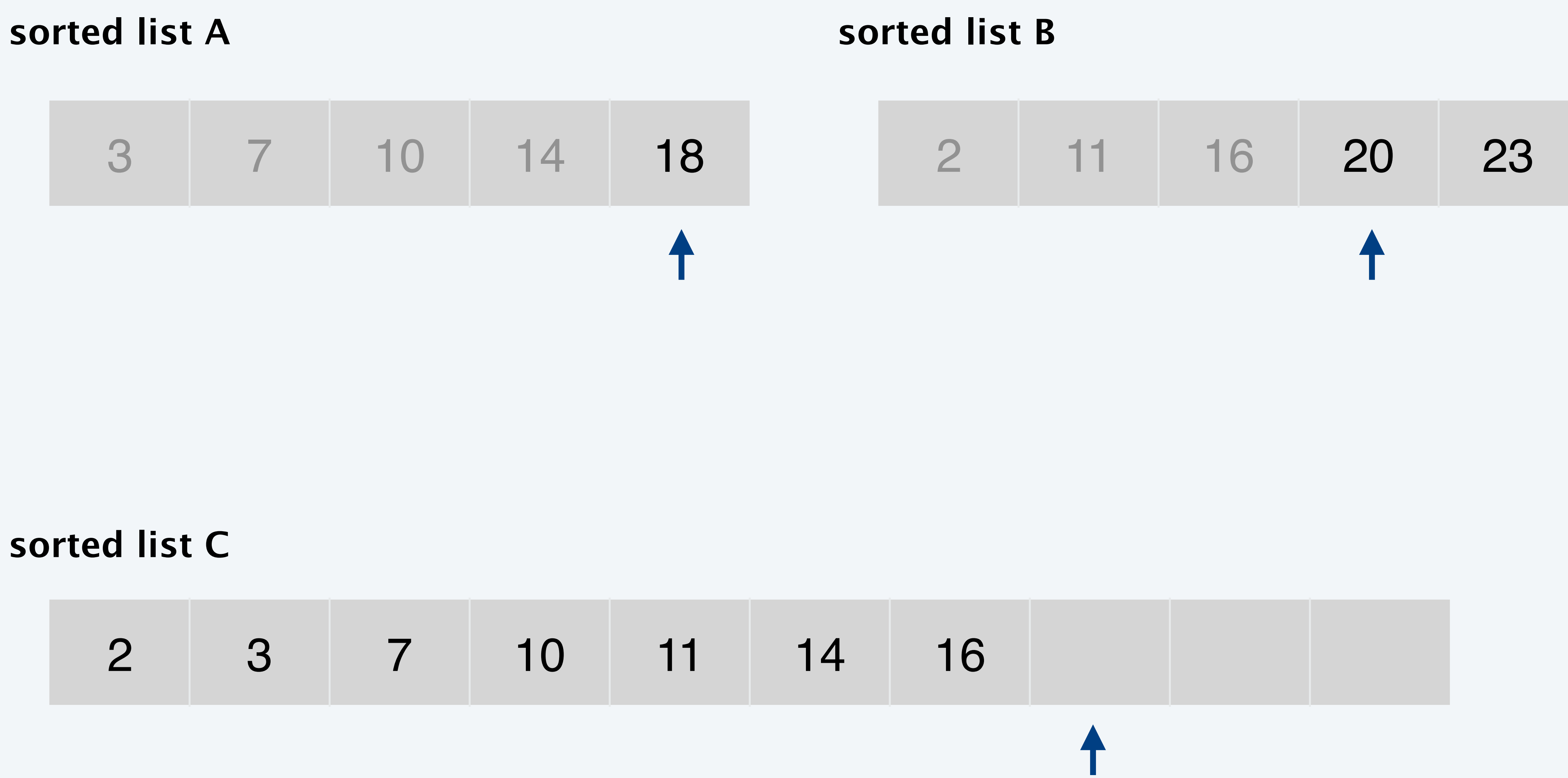

Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

compare minimum entry in each list: copy 18

sorted list C

2	3	7	10	11	14	16			

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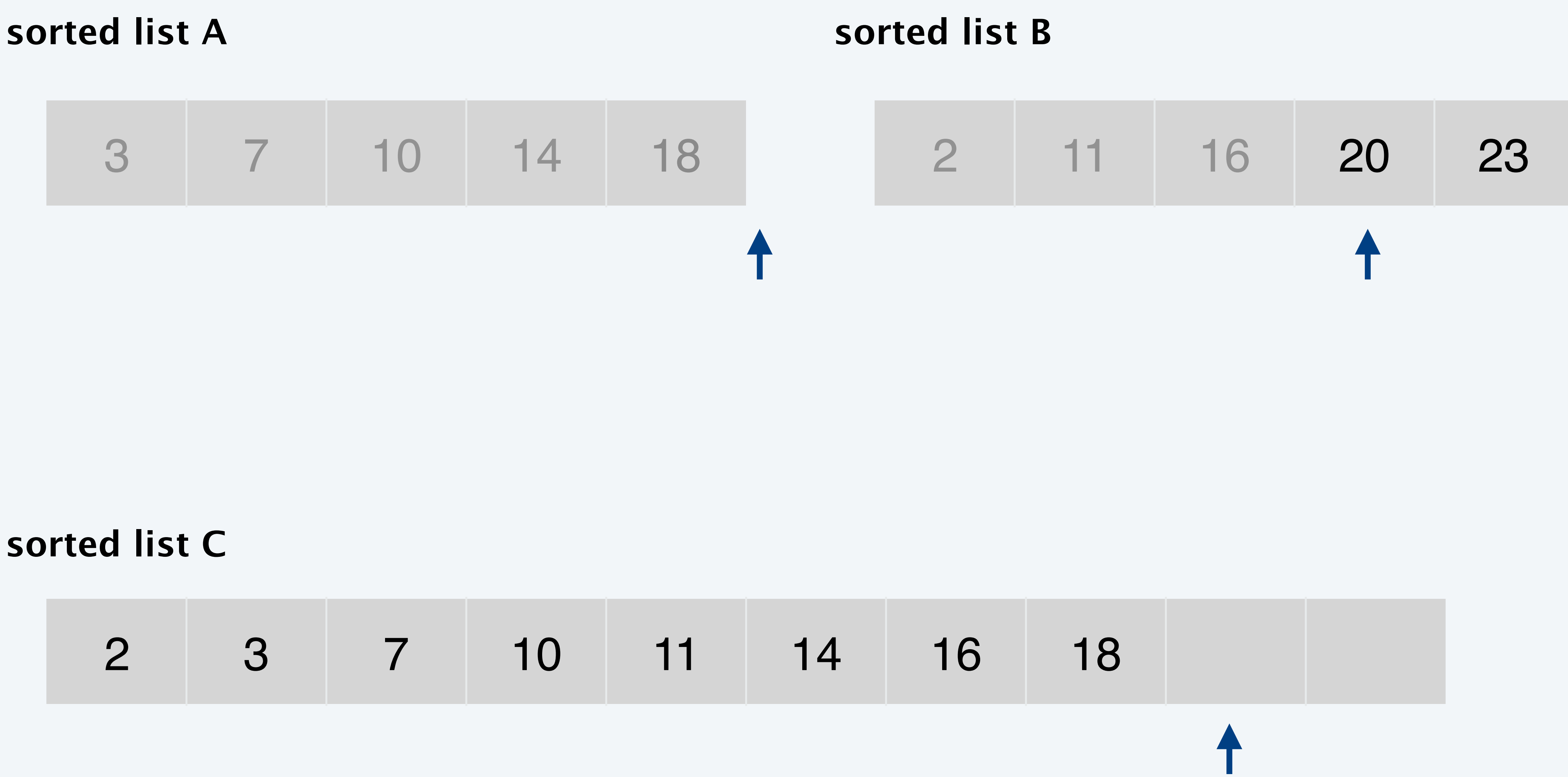

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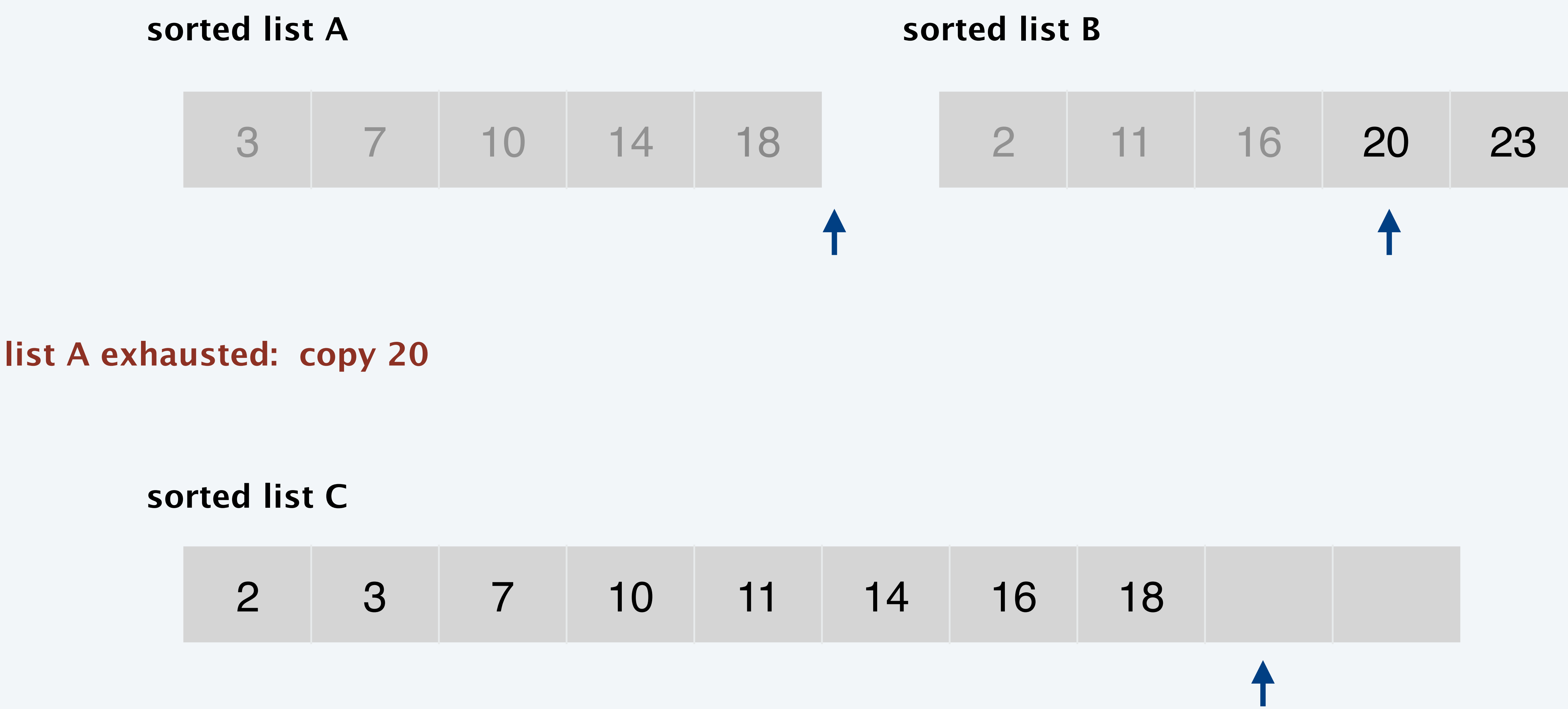

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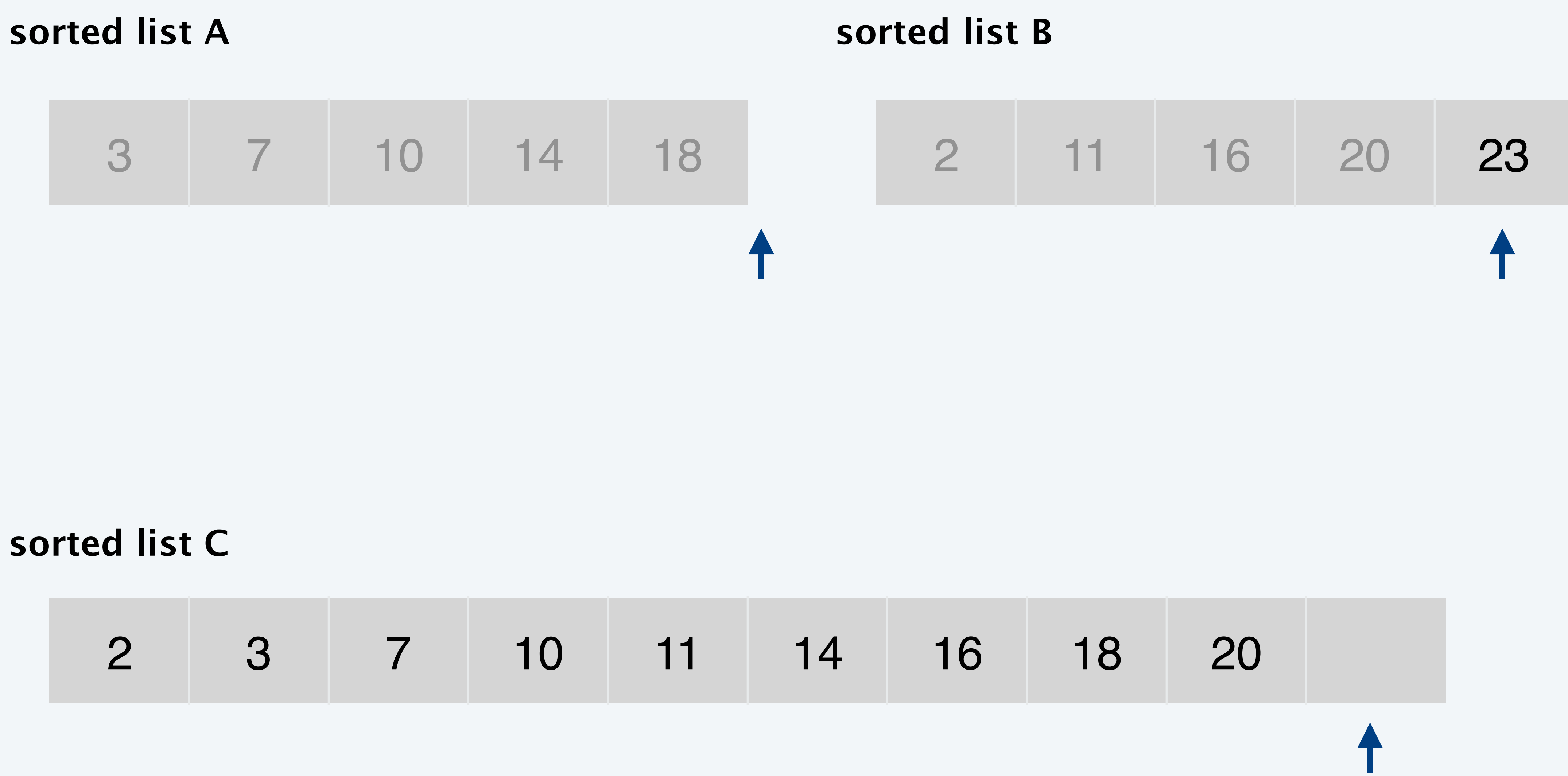

Merge demo

Given two sorted lists A and B, merge into sorted list C.

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

list A exhausted: copy 23

sorted list C

2	3	7	10	11	14	16	18	20	

Merge demo

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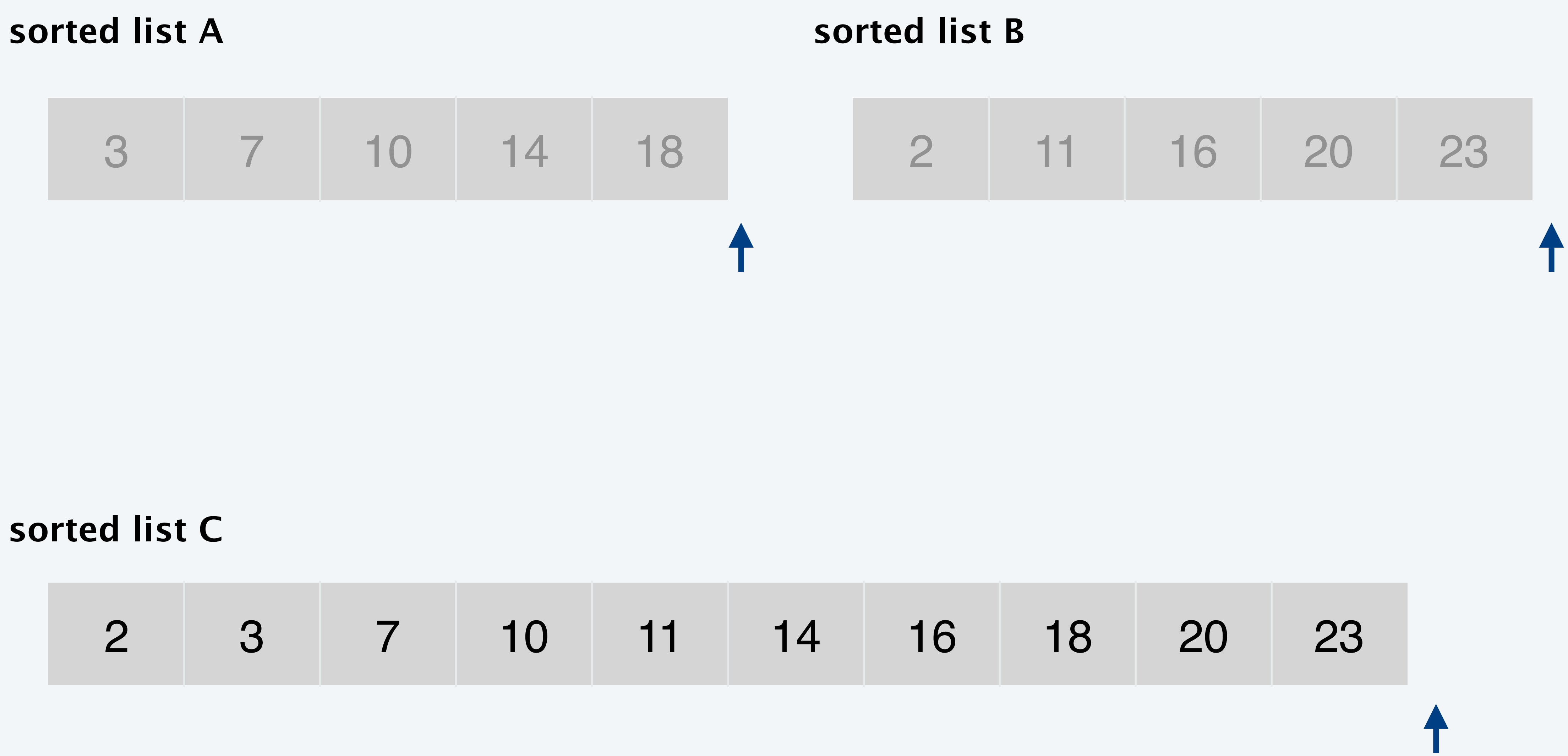

Merge demo

Is there a difference between a worst-case and best-case input for number of steps taken by this algorithm? (just the merge)

1. Yes
2. No

$\frac{n}{2}$ is $\Theta(n)$

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

1 2 3 4 5 6 7 8 9 10

sorted list C

2	3	7	10	11	14	16	18	20	23

n

$n = 10$

Merge demo

Is there a difference between a worst-case and best-case input for number of steps taken by this algorithm? Discuss with table

1. Yes
2. No

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

sorted list C

2	3	7	10	11	14	16	18	20	23

Merge demo

Is there a difference between a worst-case and best-case input for number of steps taken by this algorithm? Discuss with table

1. Yes
2. **No**

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

sorted list C

2	3	7	10	11	14	16	18	20	23

Merge demo

Is there a difference between a worst-case and best-case input for number of steps taken by this algorithm? Discuss with table: what is the runtime?

1. Yes
2. **No**

sorted list A

3	7	10	14	18

sorted list B

2	11	16	20	23

sorted list C

2	3	7	10	11	14	16	18	20	23

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.

$T(n) = 2T\left(\frac{n}{2}\right) + cn$

```
mergesort(L):
  L1 = first half of L
  L2 = first half of L
  sorted_L1 = mergesort(L1)
  sorted_L2 = mergesort(L2)
  return merged L1 and L2
```

n steps

n steps

$T(n)$ = worst-case runtime of mergesort on an input of length n

$T(n)$ = time to split list (n) + time to make 2 recursive calls ($2T(\frac{n}{2})$) + time to merge (n)

Approach # 1: unroll the recurrence

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recurrence relation

Proposition. If $T(n)$ satisfies the following recurrence, then T(n) is…

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{if } n > 1 \end{cases}$$

Approach # 1: unroll the recurrence

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{if } n > 1 \end{cases}$$

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$T(n)$

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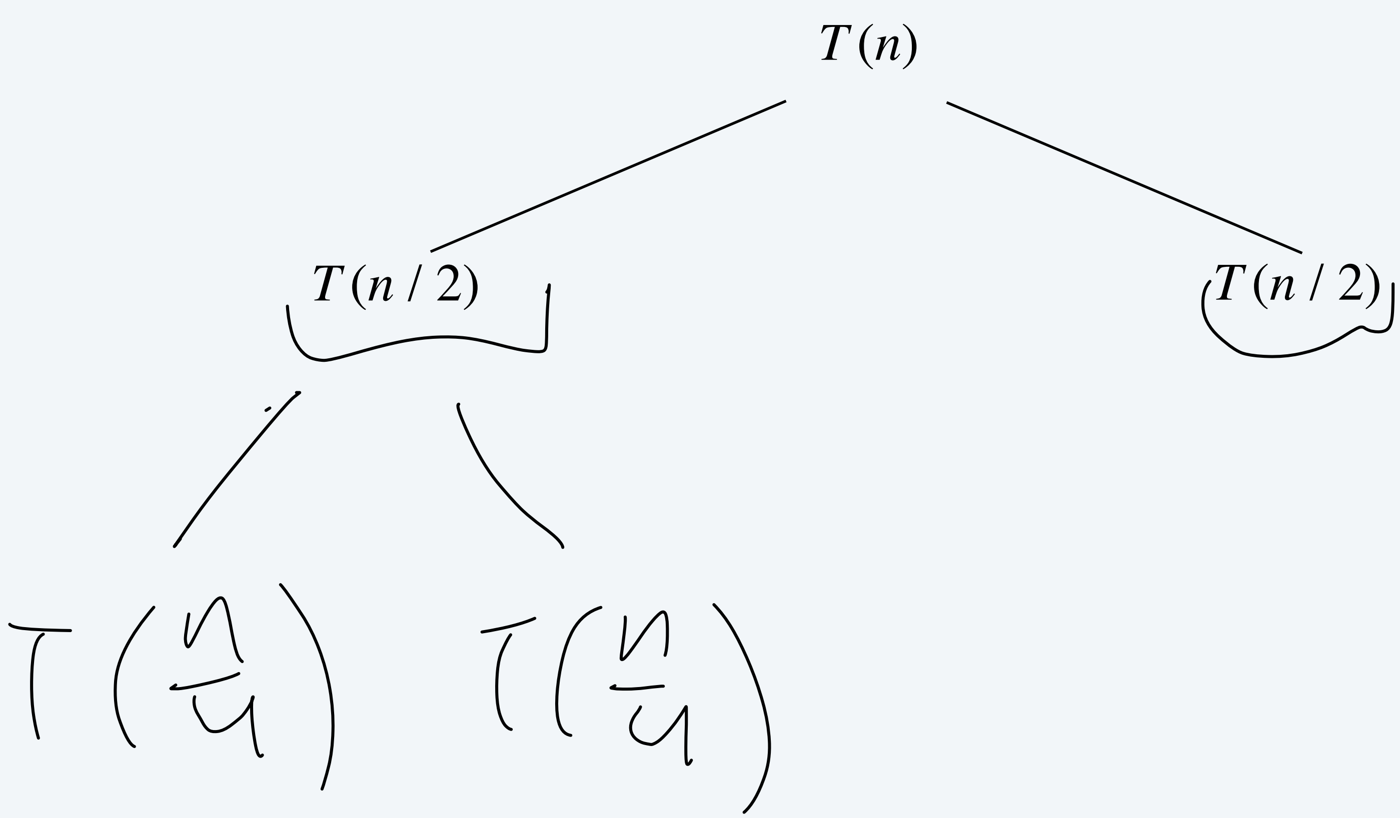

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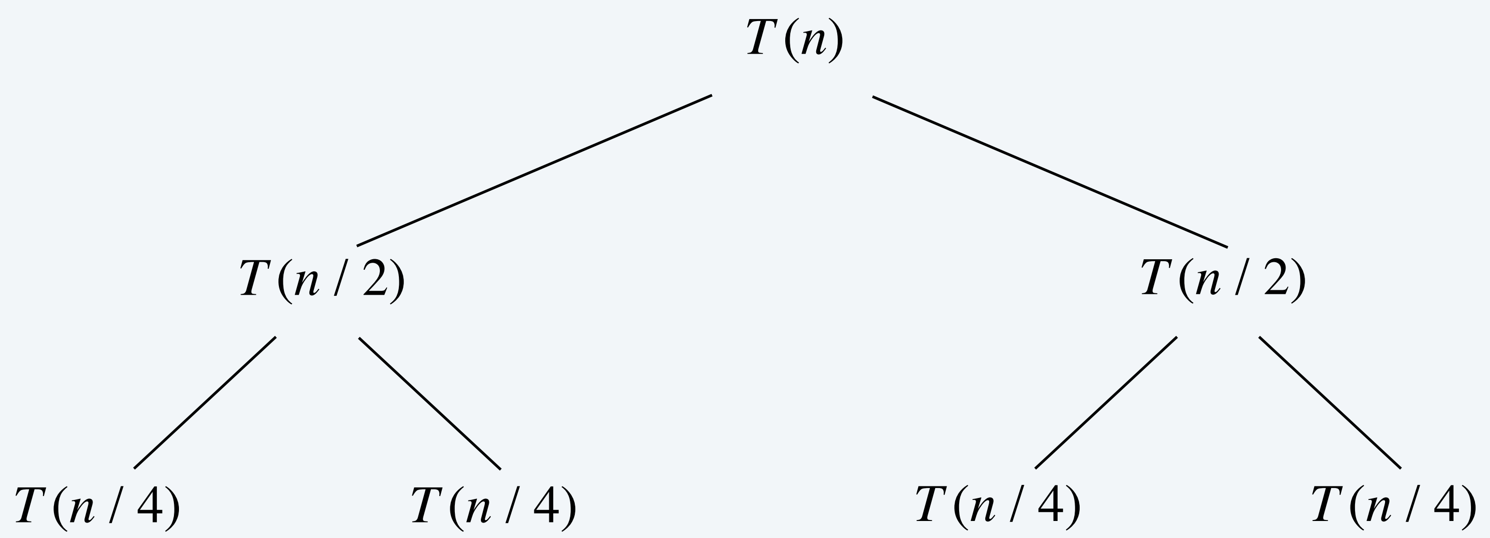

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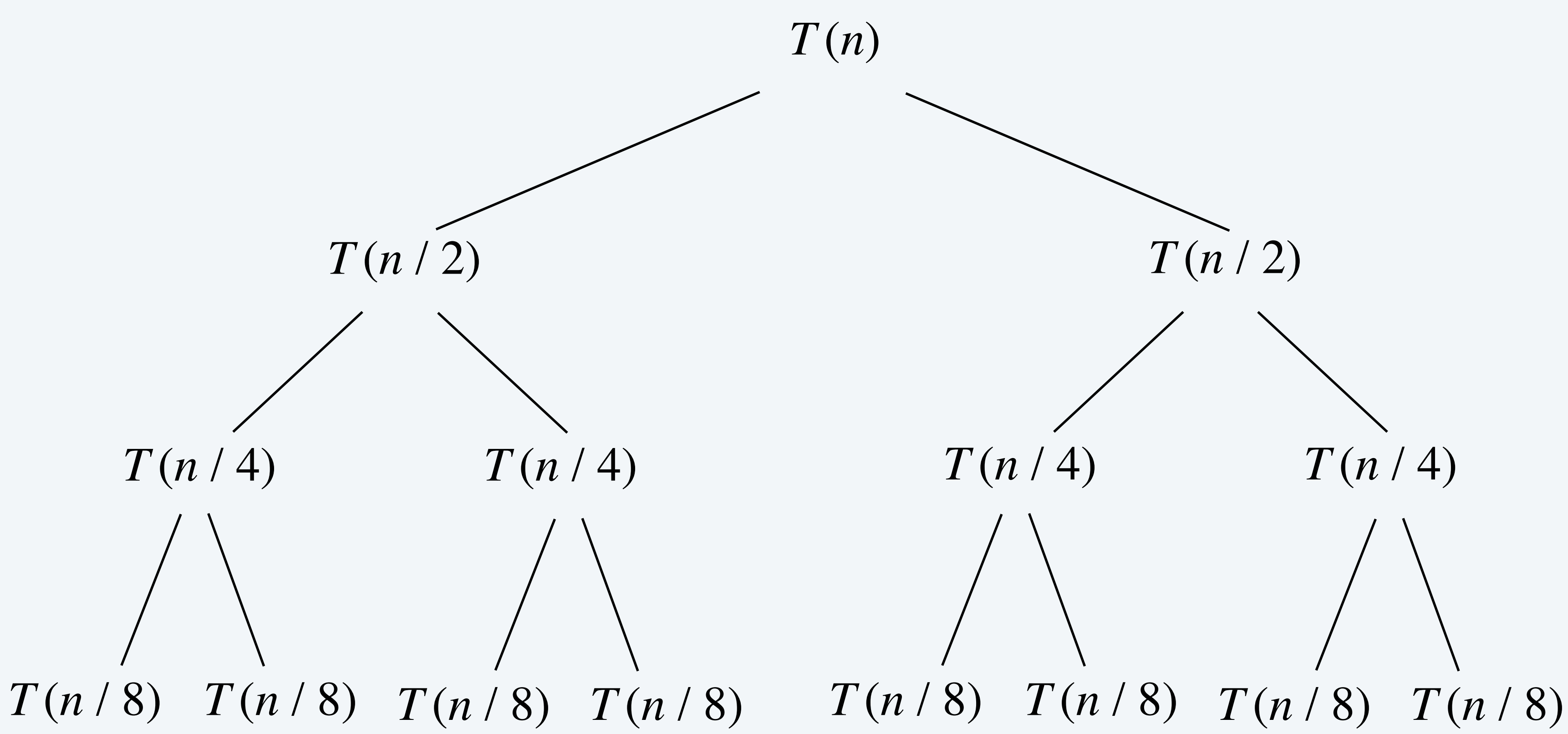

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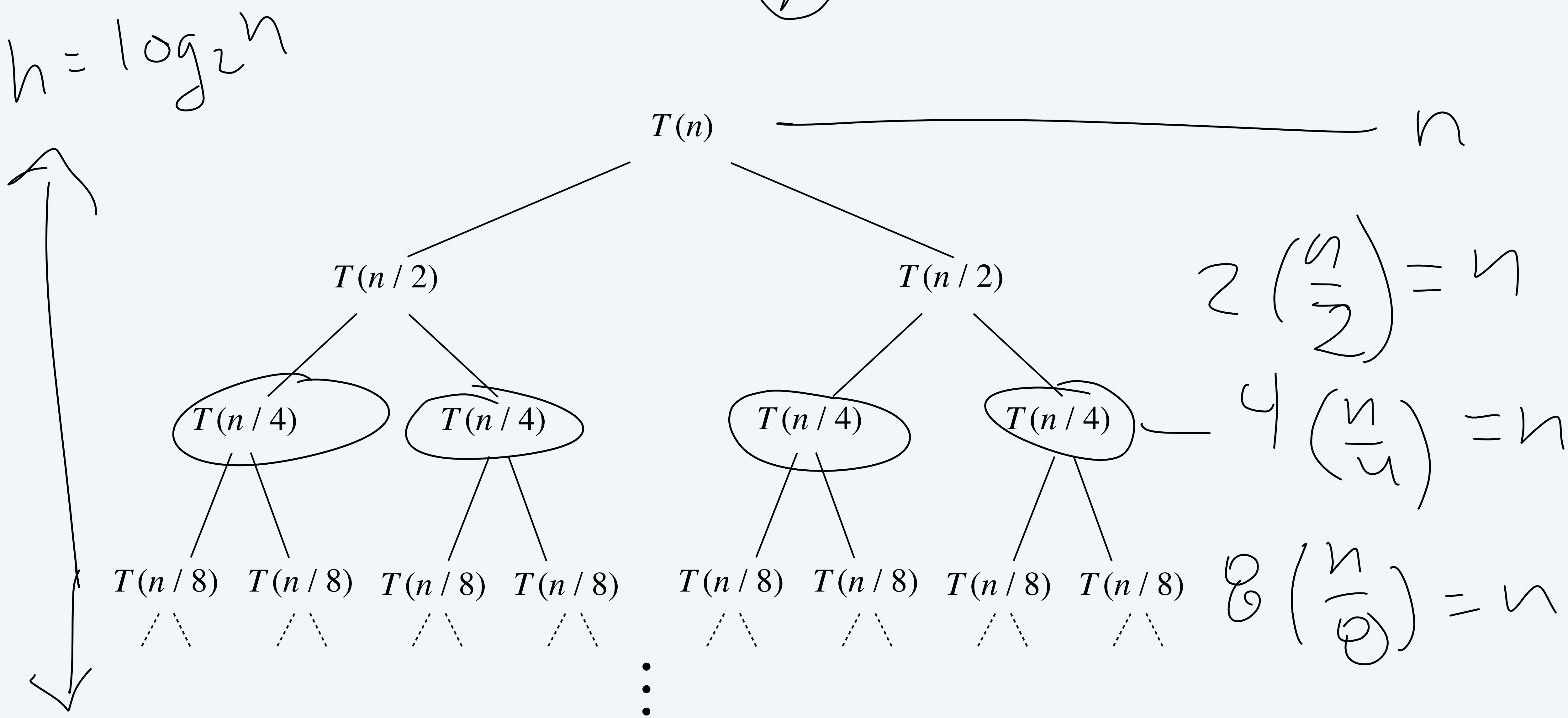

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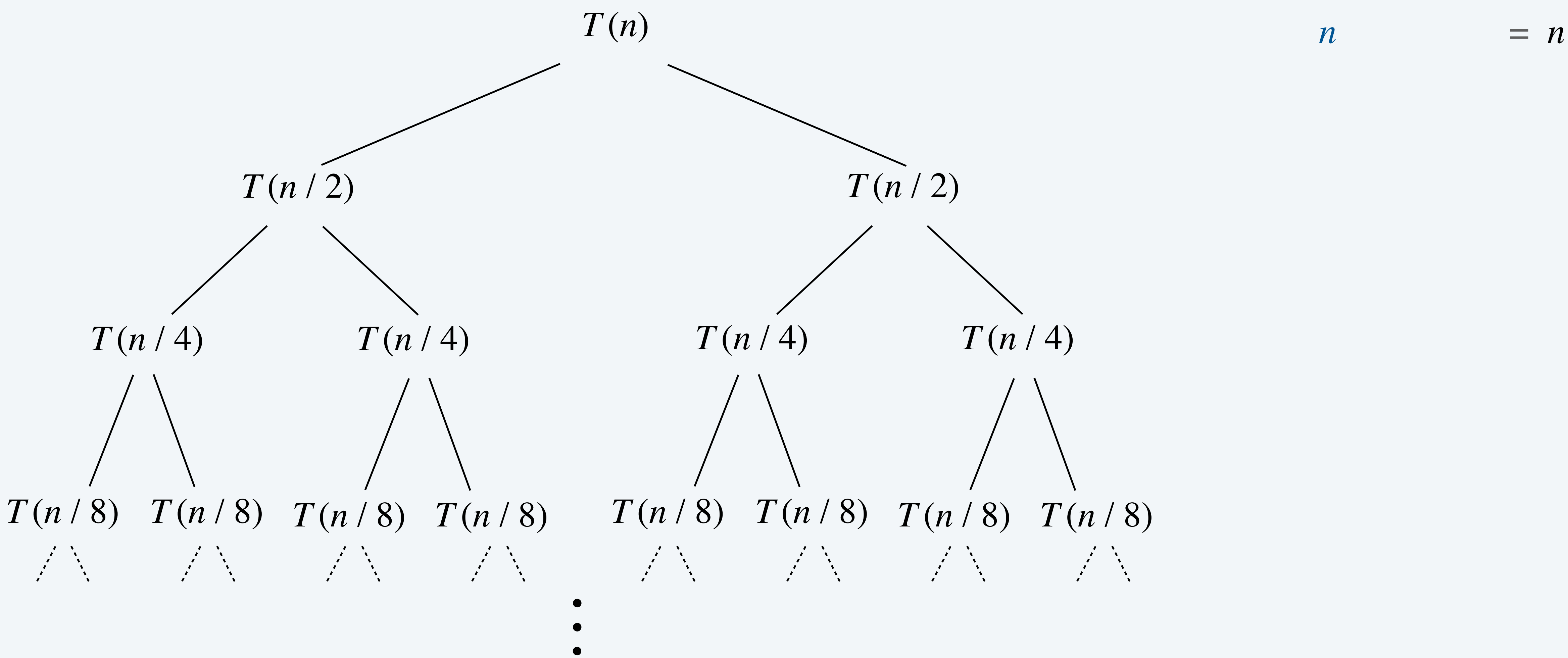

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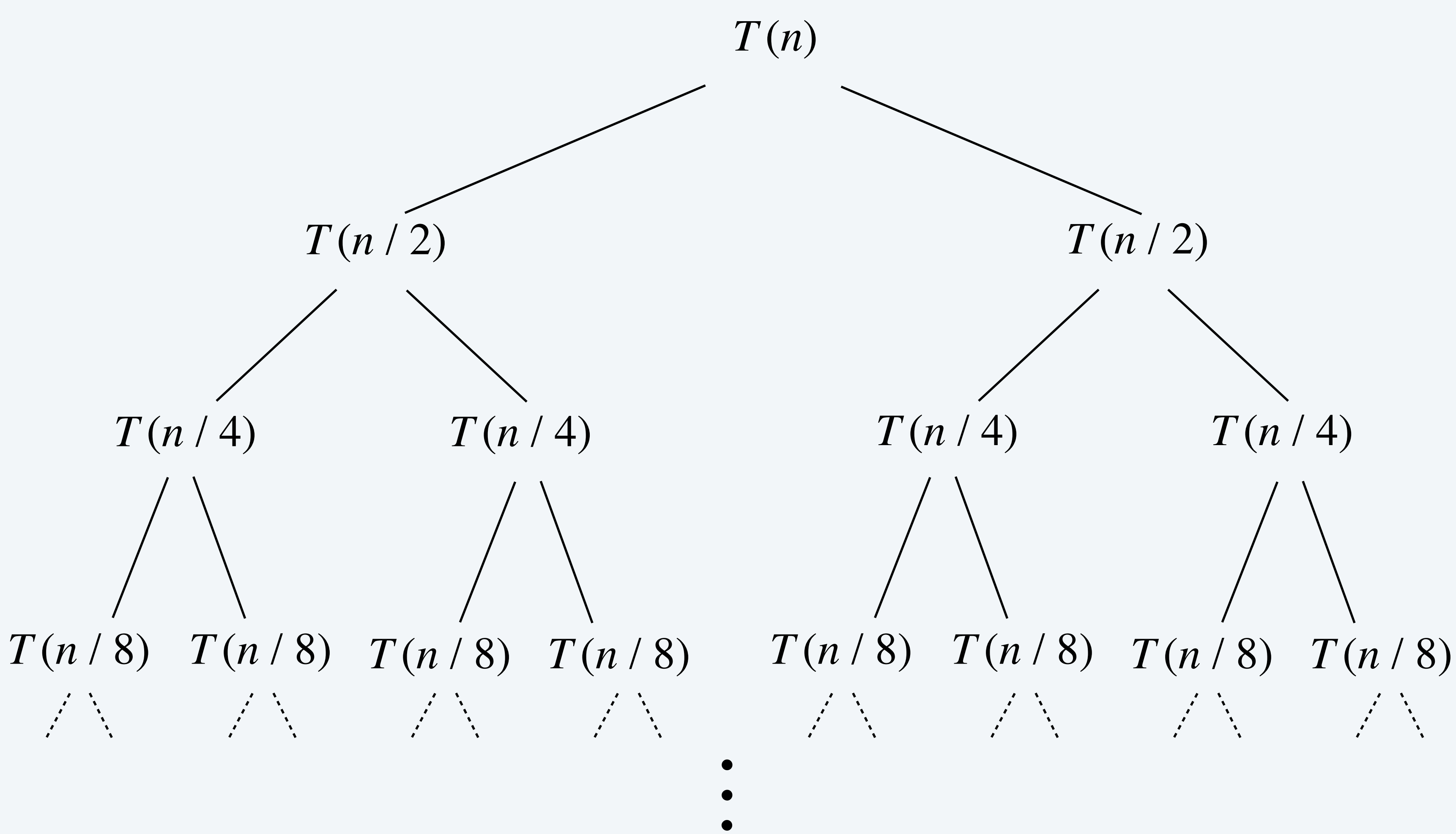

$n = n$

$2\,(n/2) = n$

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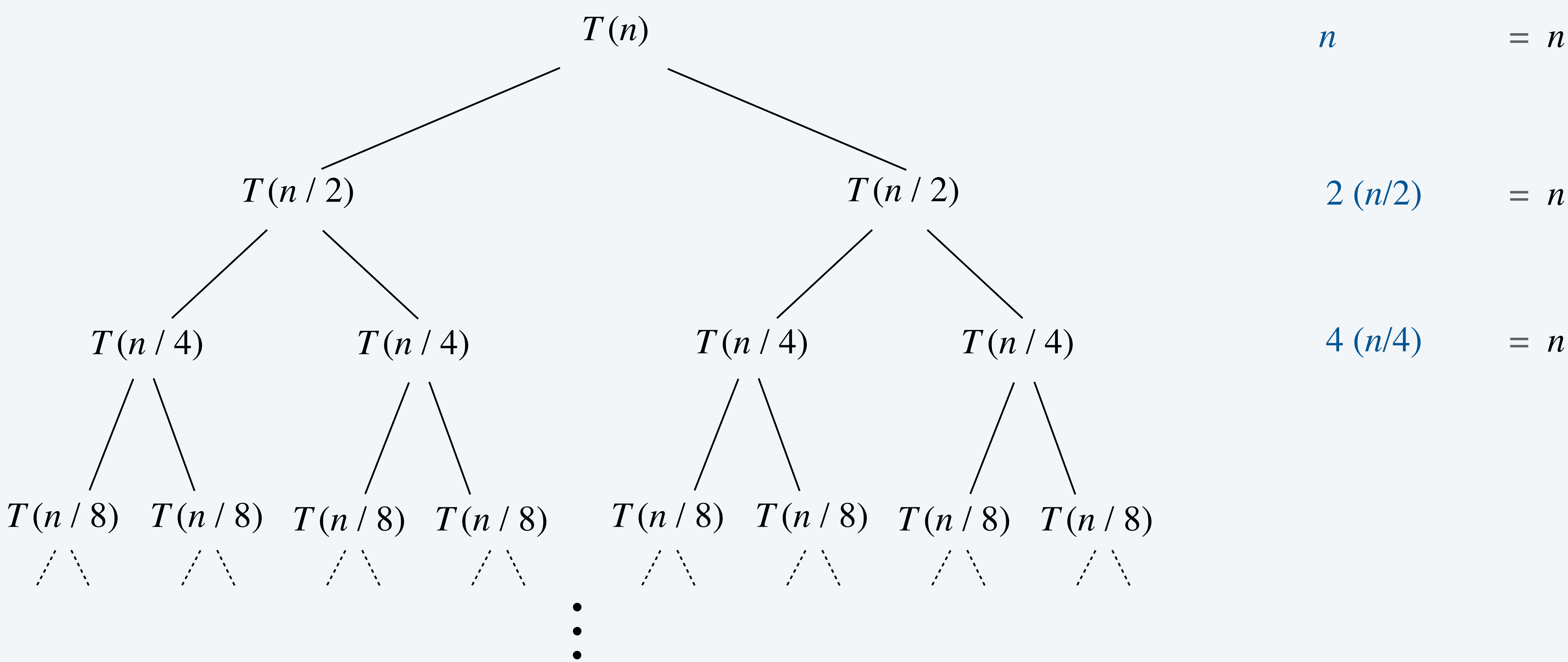

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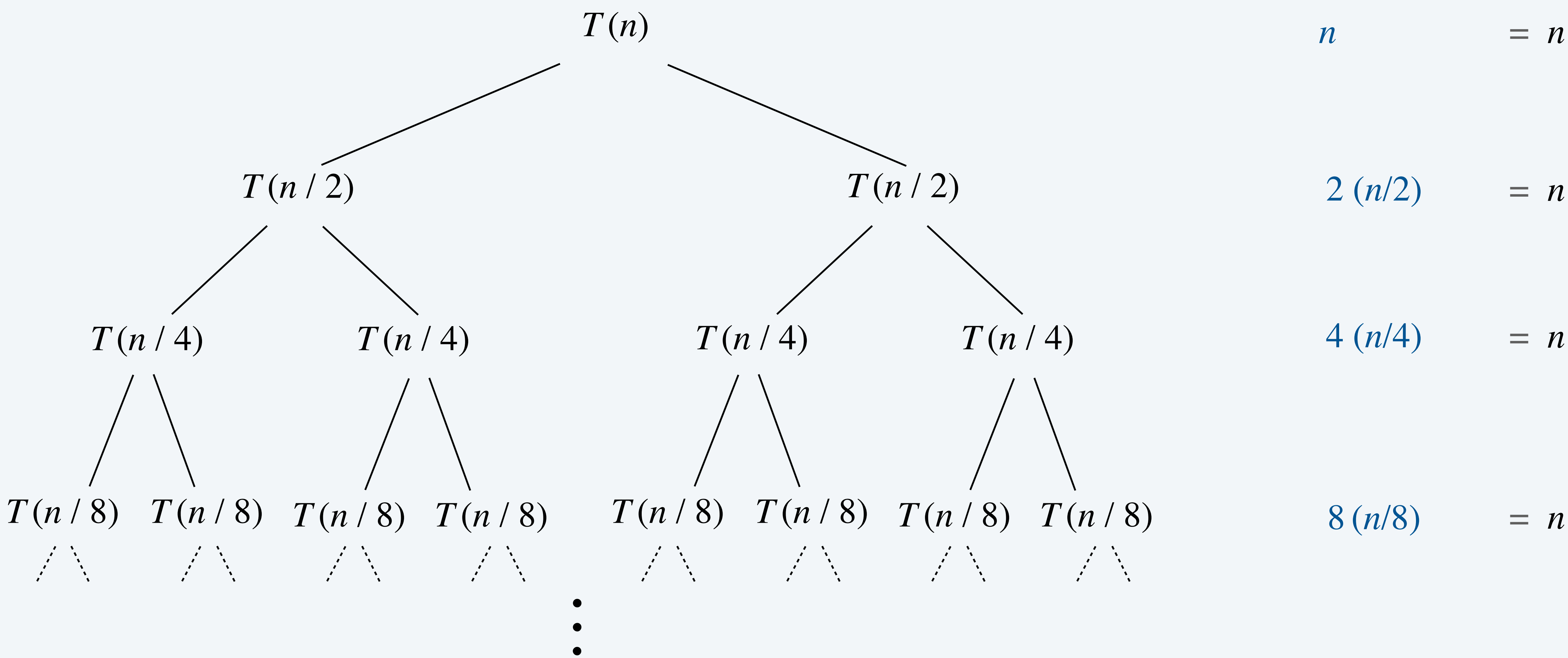

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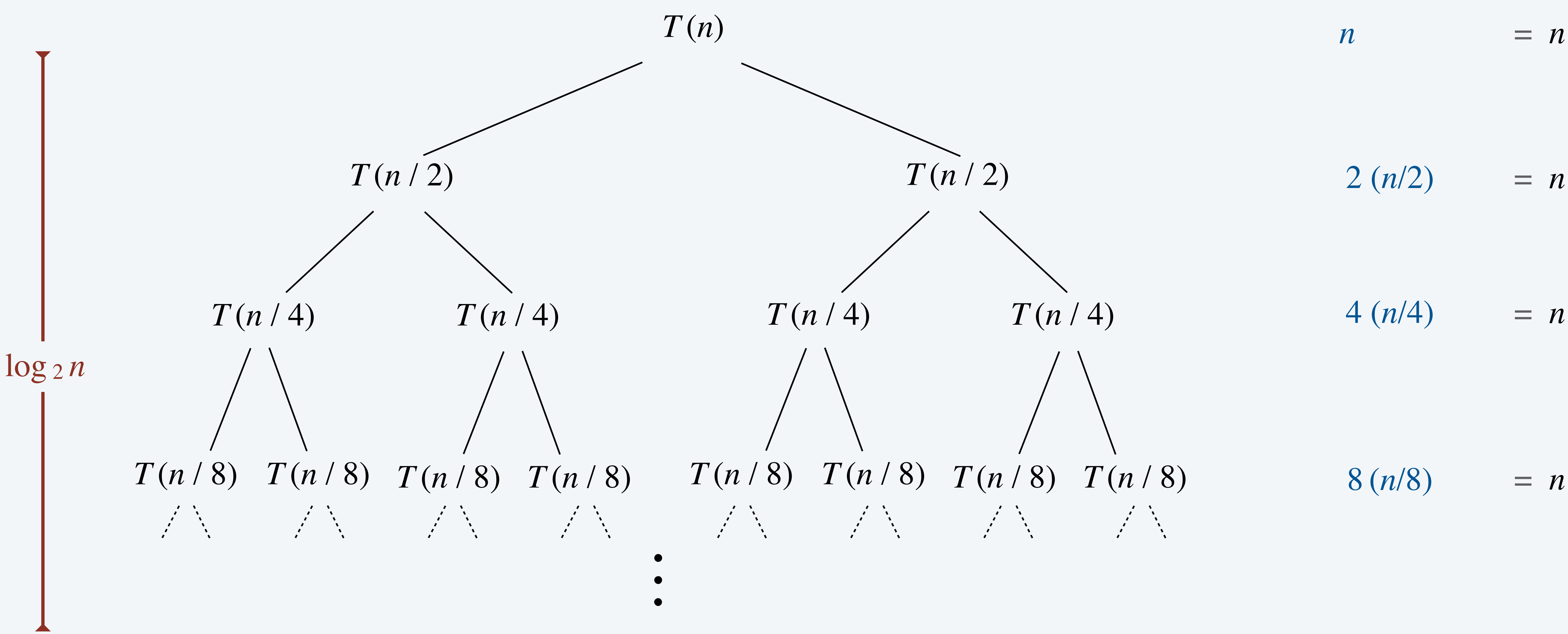

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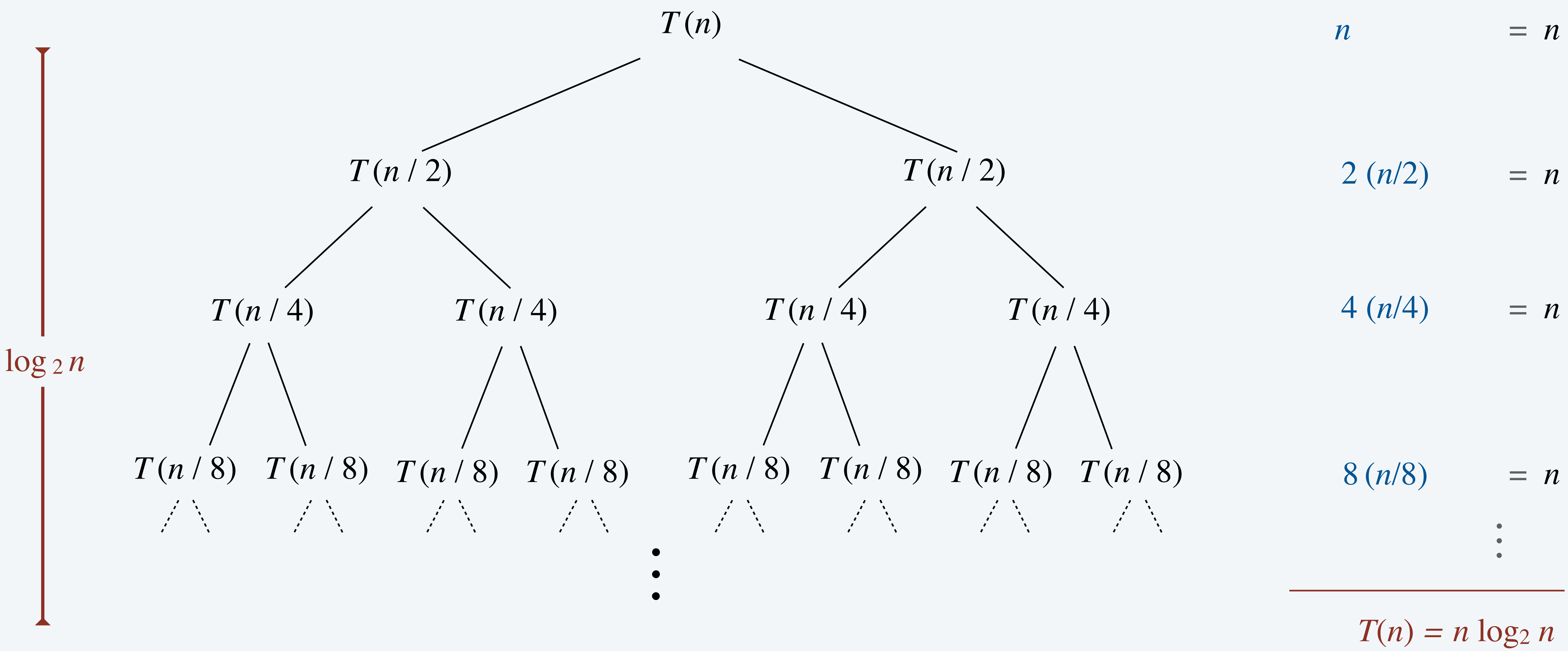

Choose an answer

```
fancymergesort(L):
  L1 = first third of L
  L2 = second third of L
  L3 = last third of L
  sorted_L1 = mergesort(L1)
  sorted_L2 = mergesort(L2)
  sorted_L3 = mergesort(L3)
  return merged L1,L2,L3
```

n

n

normal MS

$T(n) = 2T(\frac{n}{2}) + n$

3 calls to T

w/ $\frac{n}{3}$

What is a valid recurrence relation for fancymergesort?

1. $T(n) = n^2$
2. $T(n) = 3T(n/3) + n$
3. $T(n) = cn \log_3 n$
4. $T(n) = nT(n) + 3n$

$T(n) = 3T(\frac{n}{3}) + n$

Proof by induction

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{if } n > 1 \end{cases}$$

assuming n is a power of 2

Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{if } n > 1 \end{cases}$$

Claim: for $n \geq 1$, if [above] then $T(n) = n \log n$

Proof:

Let n be ≥ 1.

Assume that for all $m < n$, if $T(m) = 2T(m/2) + m$ (for $m > 1$) then $T(m) = m \log m$.

Base case: If $n = 1$, then $T(1) = 0$. And $1 \cdot \log 1 = 0$.

If $n > 1$, then $T(n) = 2T(n/2) + n$

$= 2\left(\frac{n}{2} \log \frac{n}{2}\right) + n$ by IH

$= n \log \frac{n}{2} + n$

$$= n\left(\log \frac{n}{2} + 1\right)$$

$$= n\left(\log_2 \frac{n}{2} + \log_2 2\right)$$

$$= n\left(\log_2 \cancel{2}\left(\frac{n}{\cancel{2}}\right)\right)$$

$$= n \log n$$

Digression: sorting lower bound

Challenge. How to prove a lower bound for all conceivable algorithms?

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Model of computation. Comparison trees.

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- Can access the elements only through pairwise comparisons.

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- Can access the elements only through pairwise comparisons.
- All other operations (control, data movement, etc.) are free.

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Cost model. Number of compares.

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Cost model. Number of compares.

Q. Realistic model?

A1. Yes. Java, Python, C++, …

```
Comparable[] a = ...;
...
Arrays.sort(a);
```

← can access elements only via calls to compareTo()

Digression: sorting lower bound

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`sort`(***, *key=None*, *reverse=False*)

This method sorts the list in place, using only `<` comparisons between items. Exceptions are not suppressed – if any comparison operations fail, the entire sort operation will fail (and the list will likely be left in a partially modified state).

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Challenge. How to prove a lower bound for all conceivable algorithms?

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A1. Yes. Java, Python, C++, …

A2. Yes. Mergesort, insertion sort, quicksort, heapsort, …

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Model of computation. Comparison trees.

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Cost model. Number of compares.

Q. Realistic model?

A1. Yes. Java, Python, C++, …

A2. Yes. Mergesort, insertion sort, quicksort, heapsort, …

A3. No. Bucket sort, radix sorts, …

Comparison tree (for 3 distinct keys a, b, and c)

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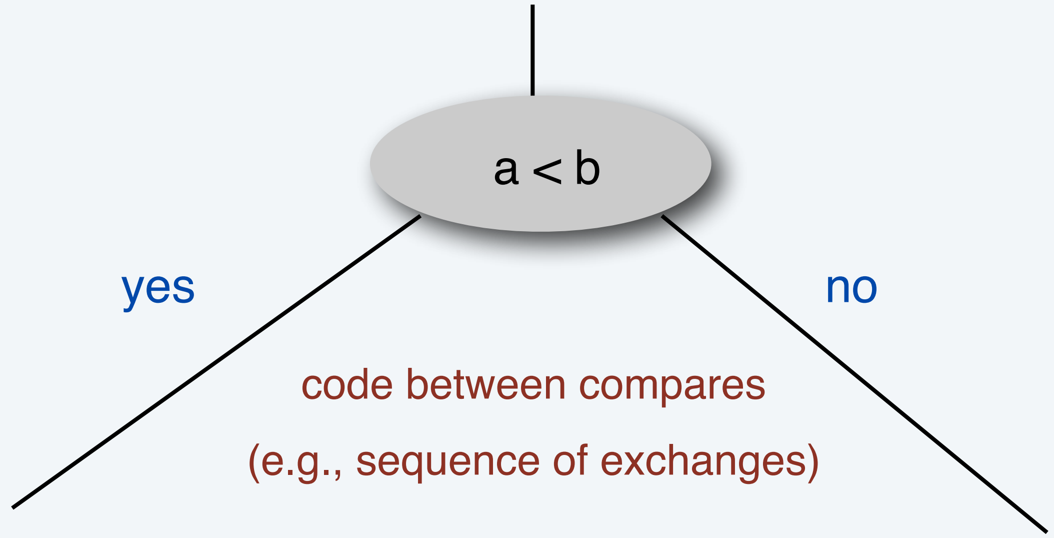

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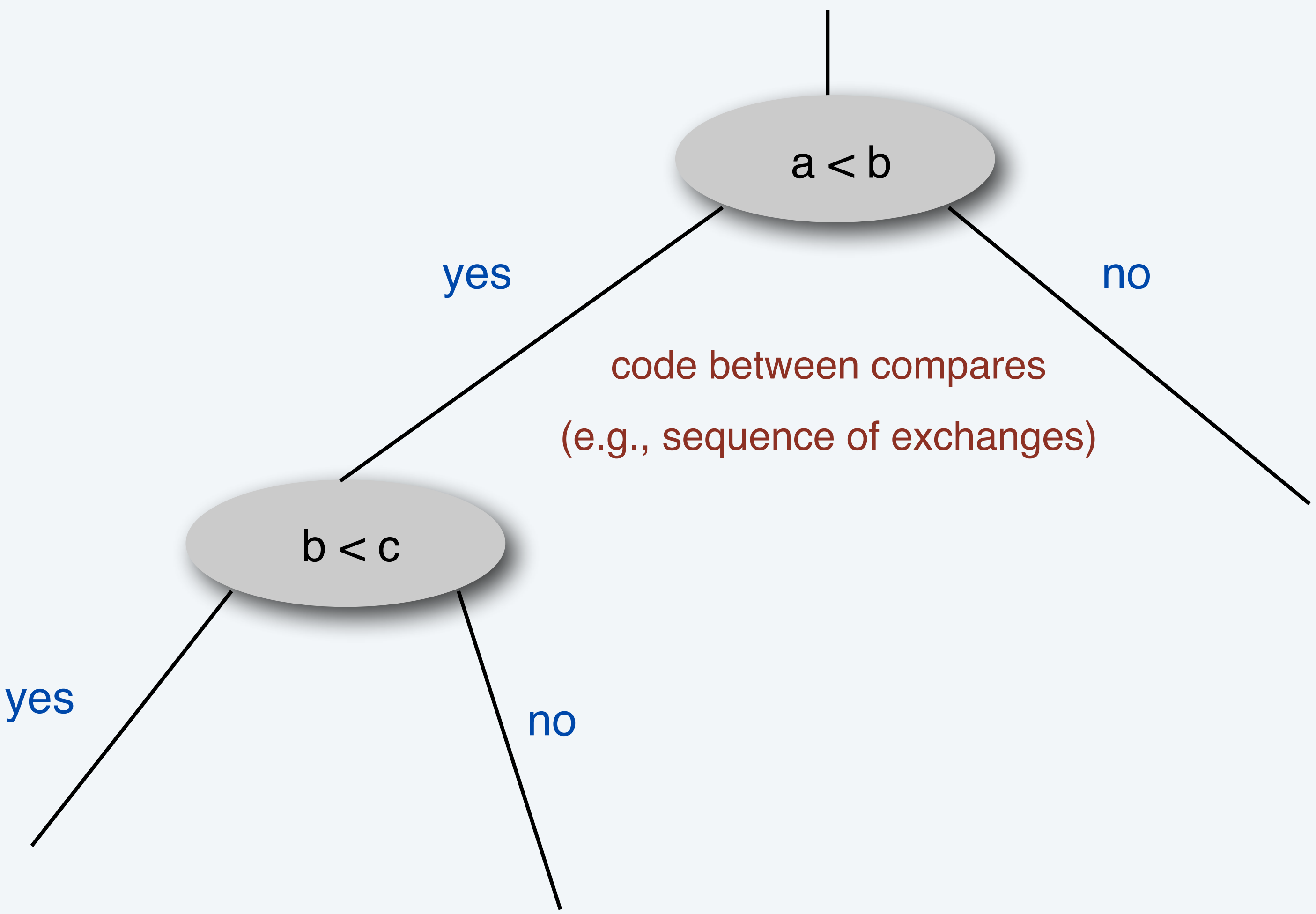

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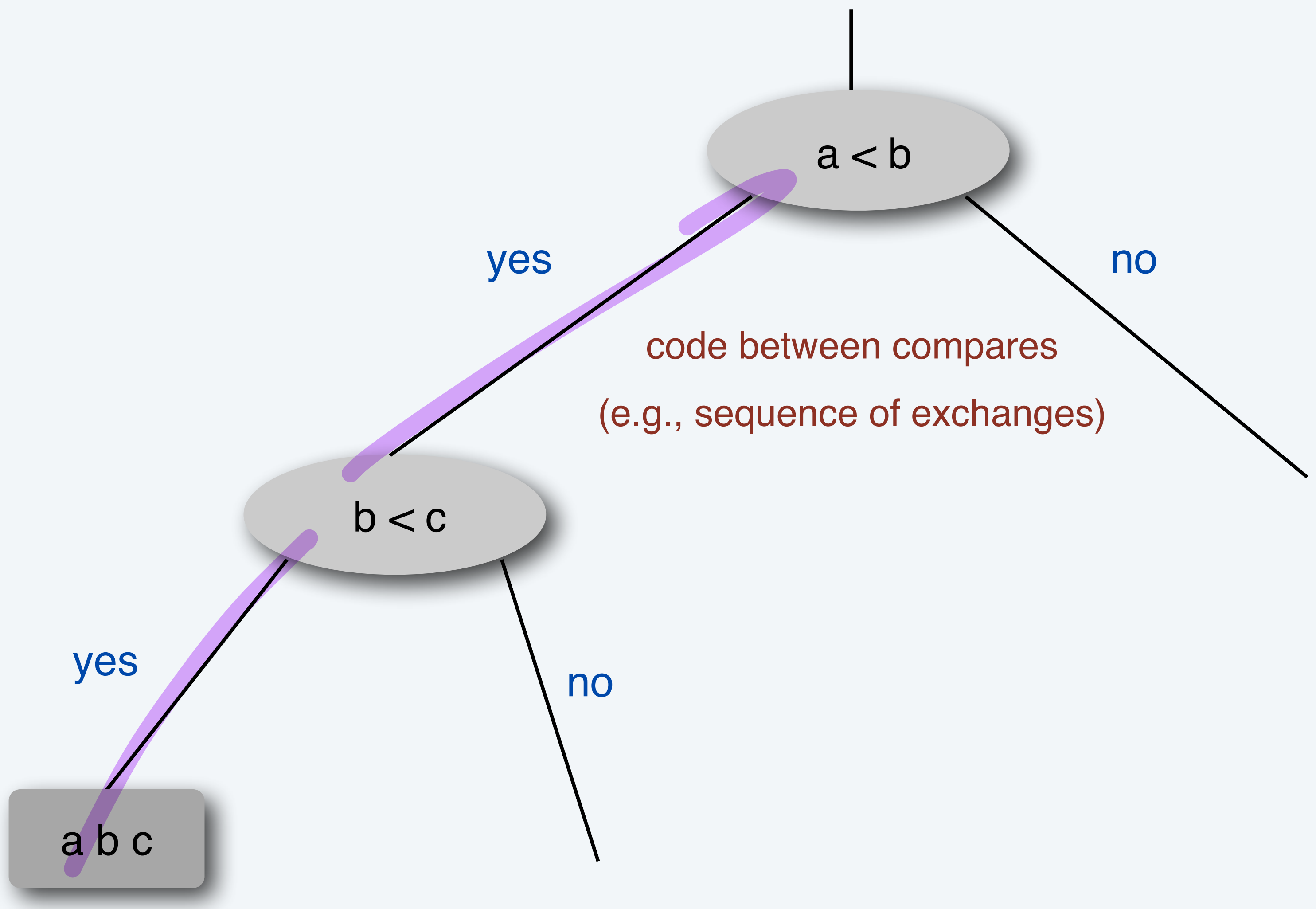

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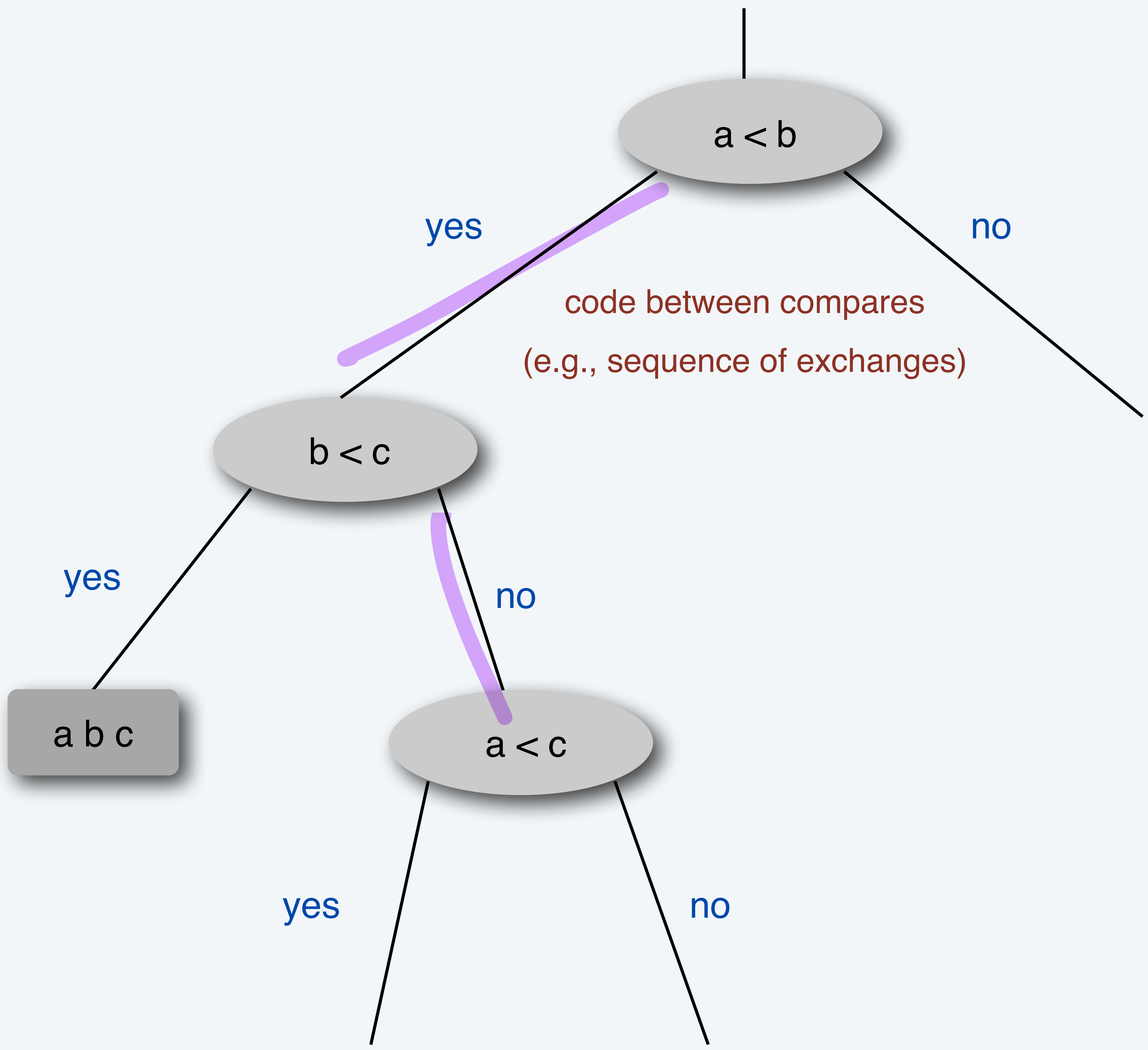

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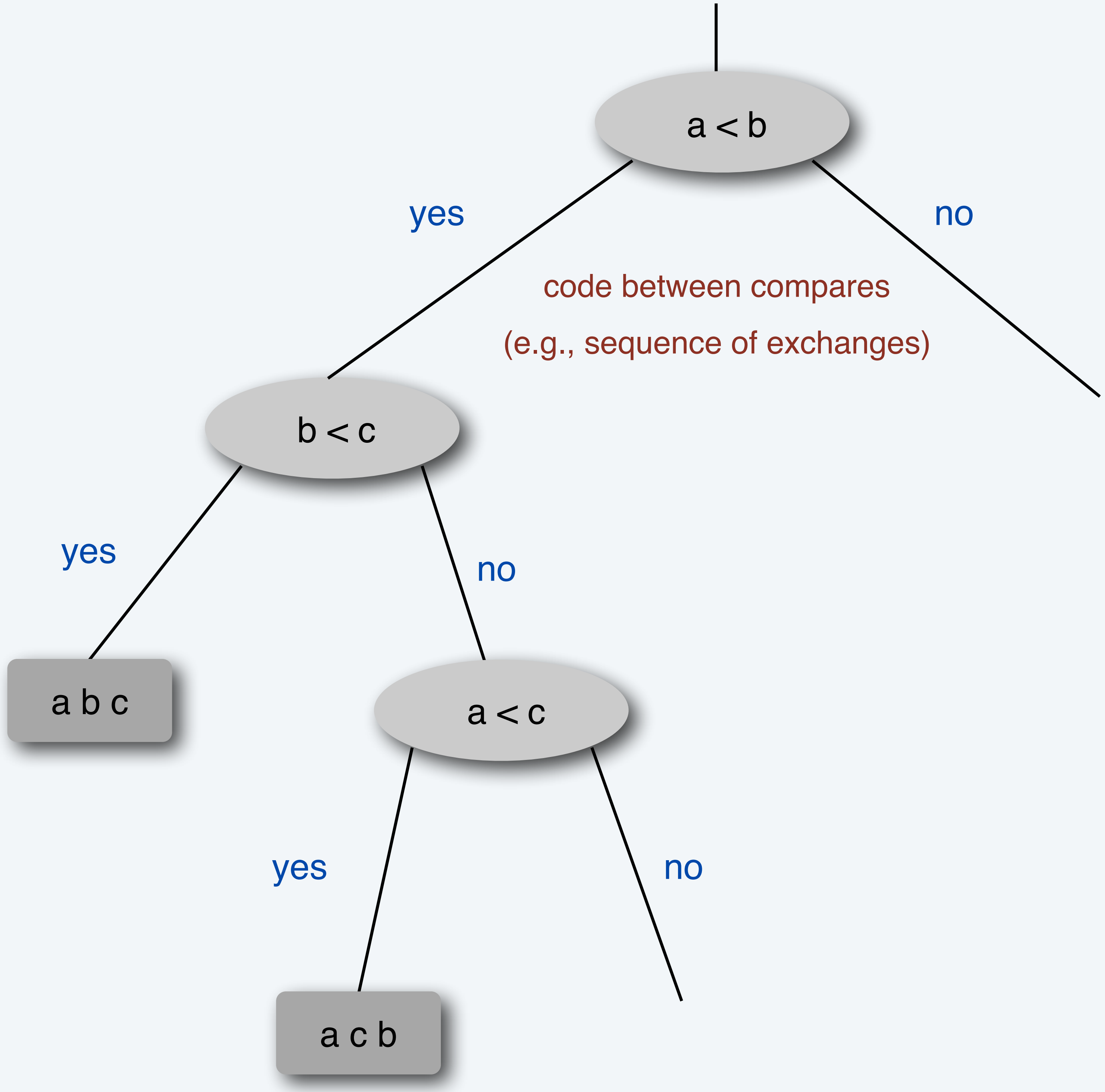

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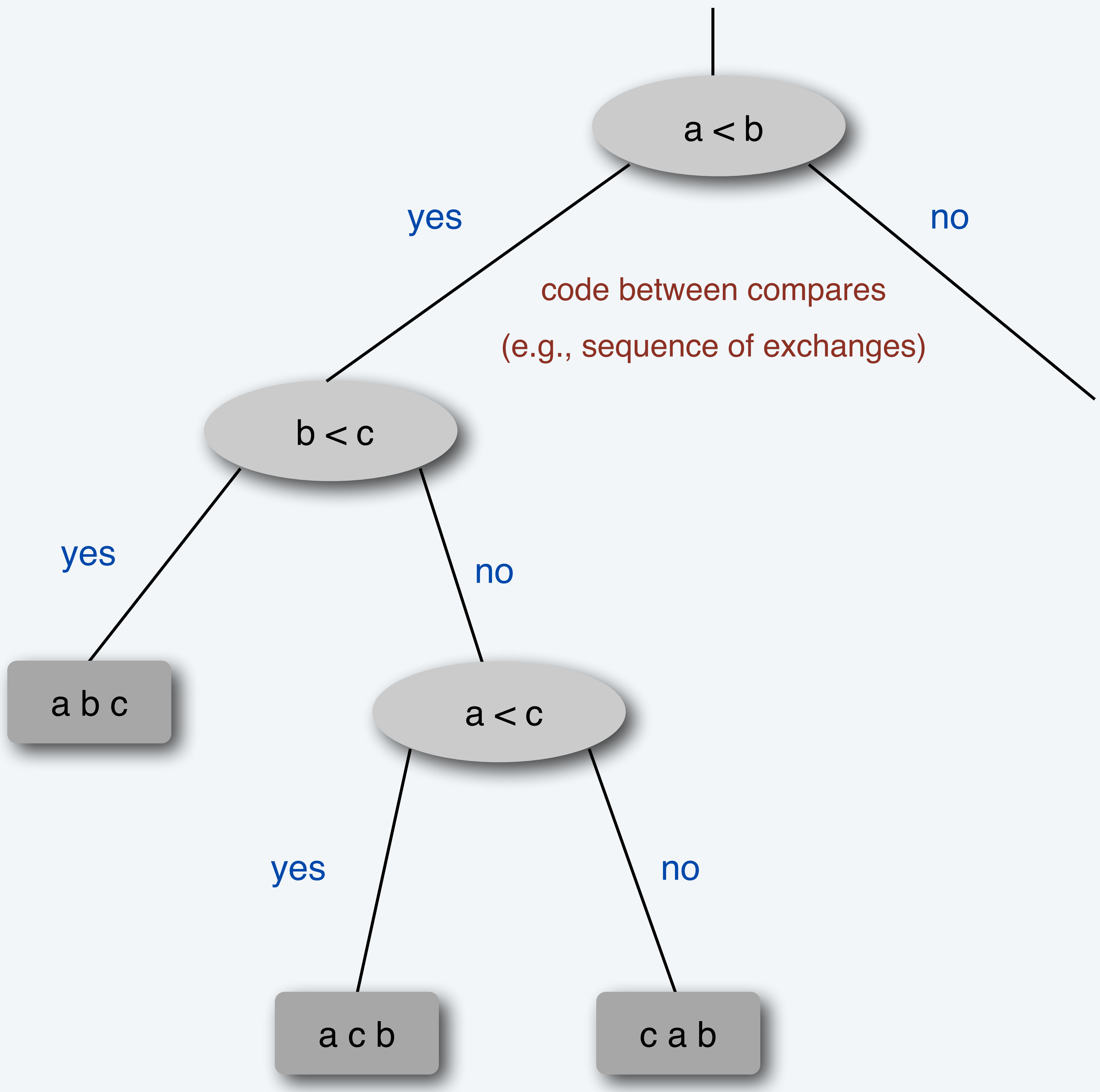

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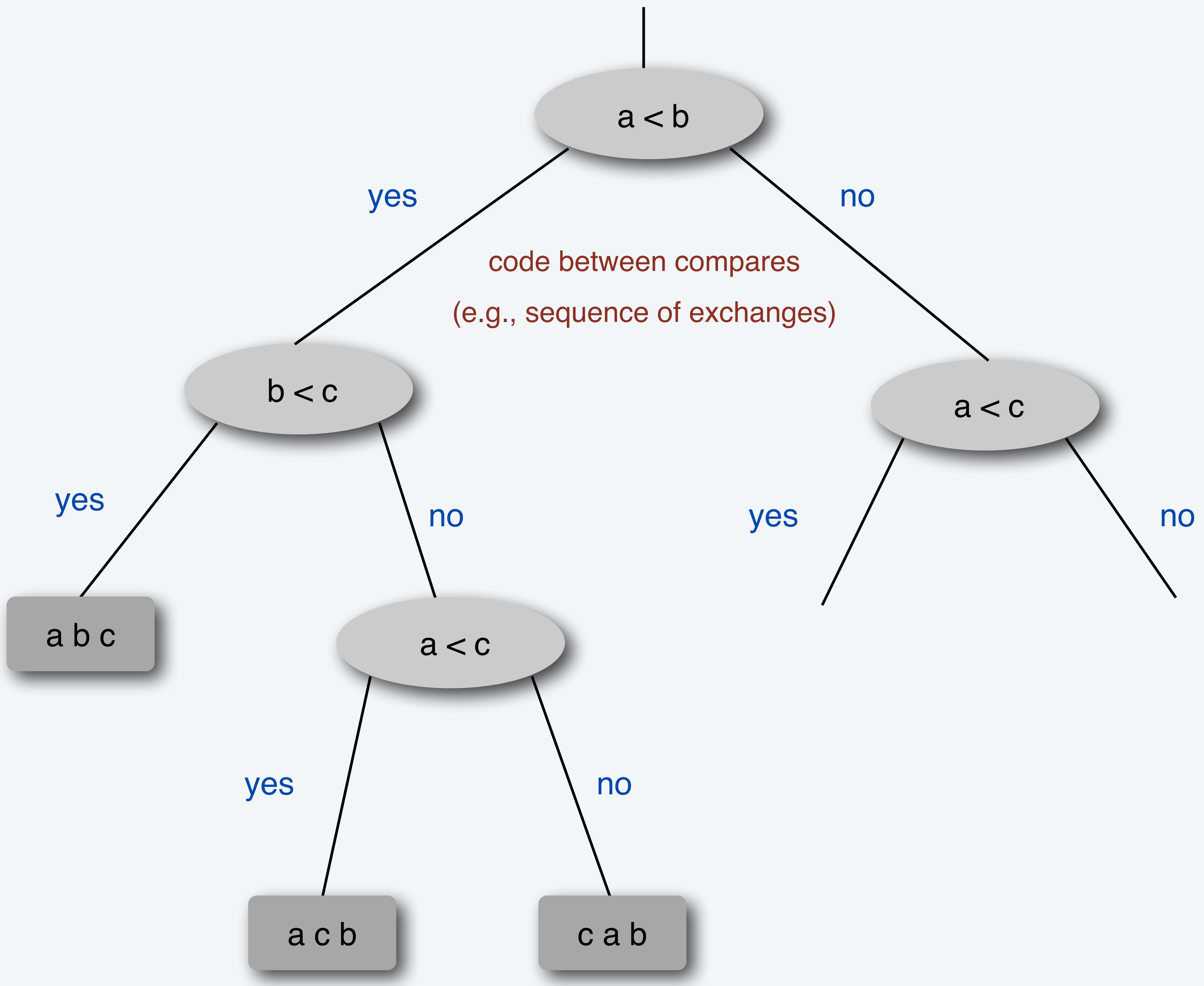

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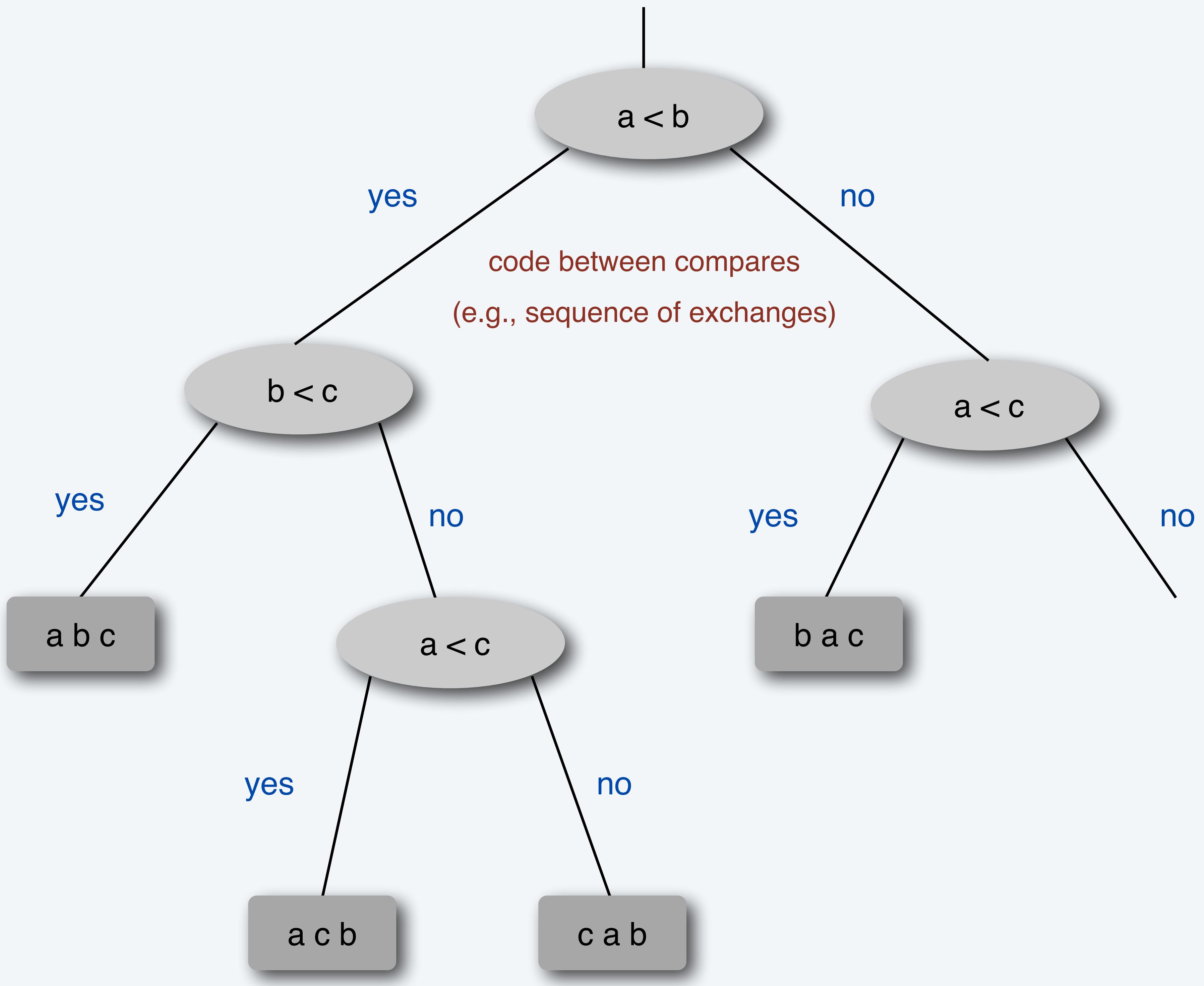

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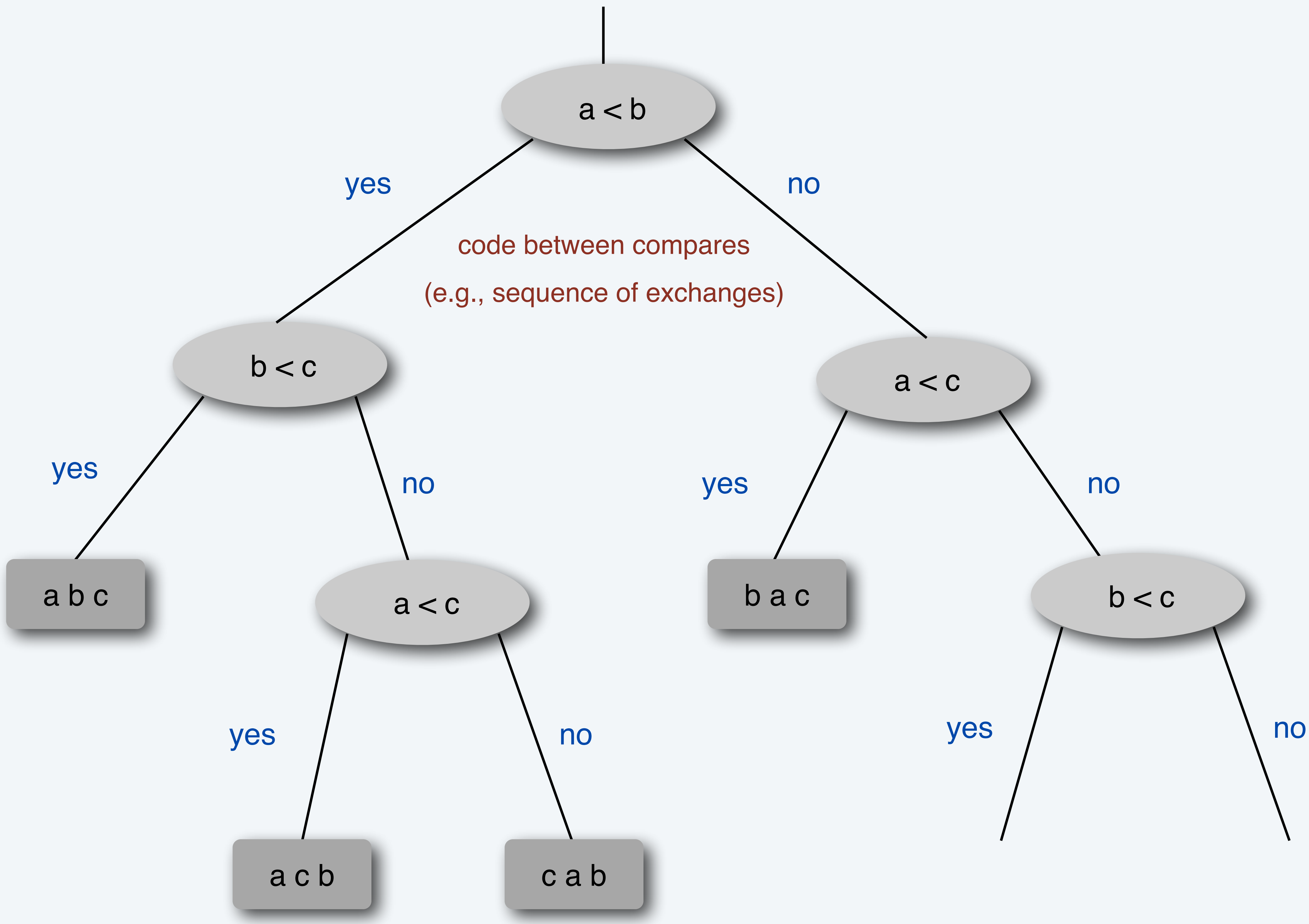

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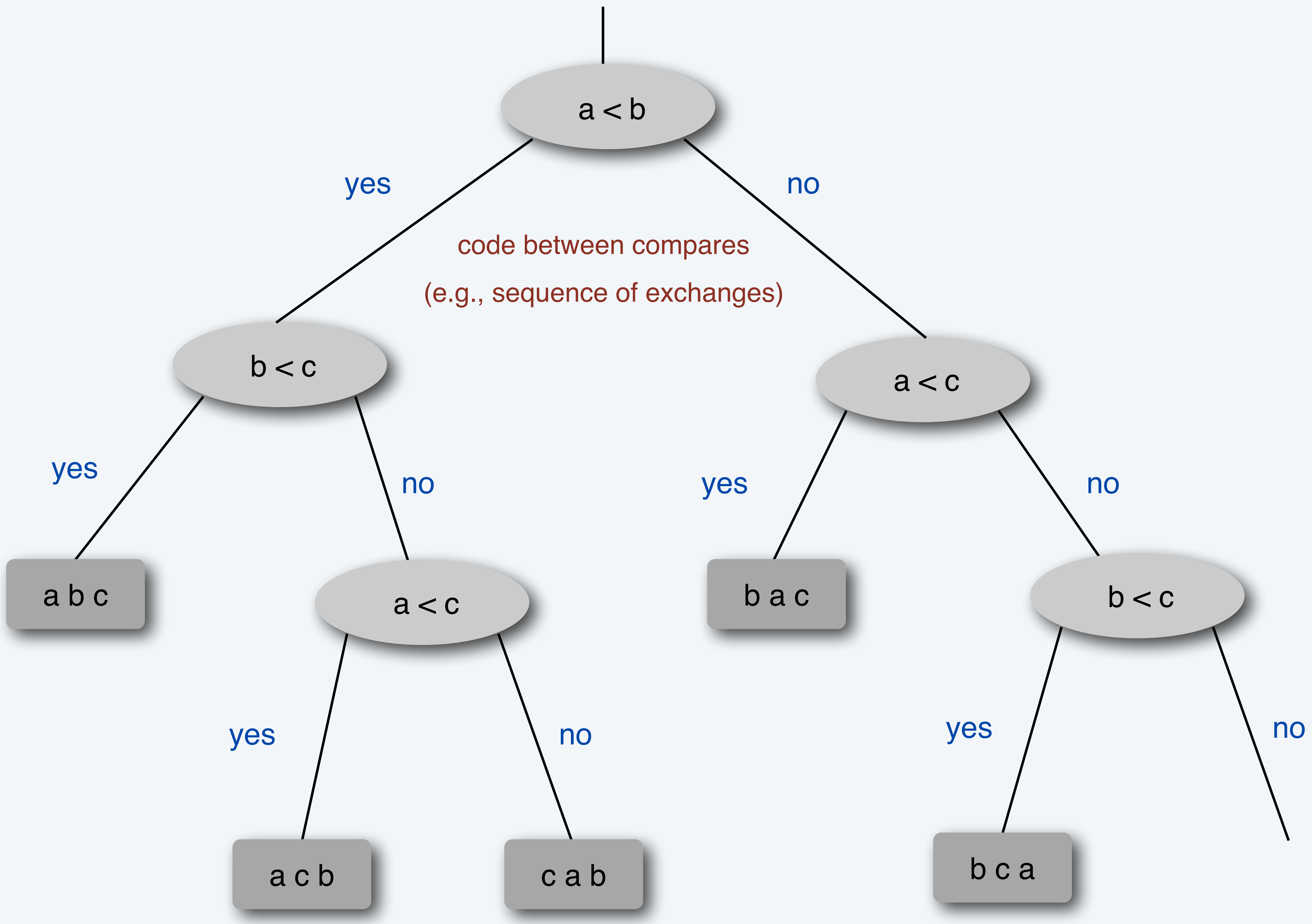

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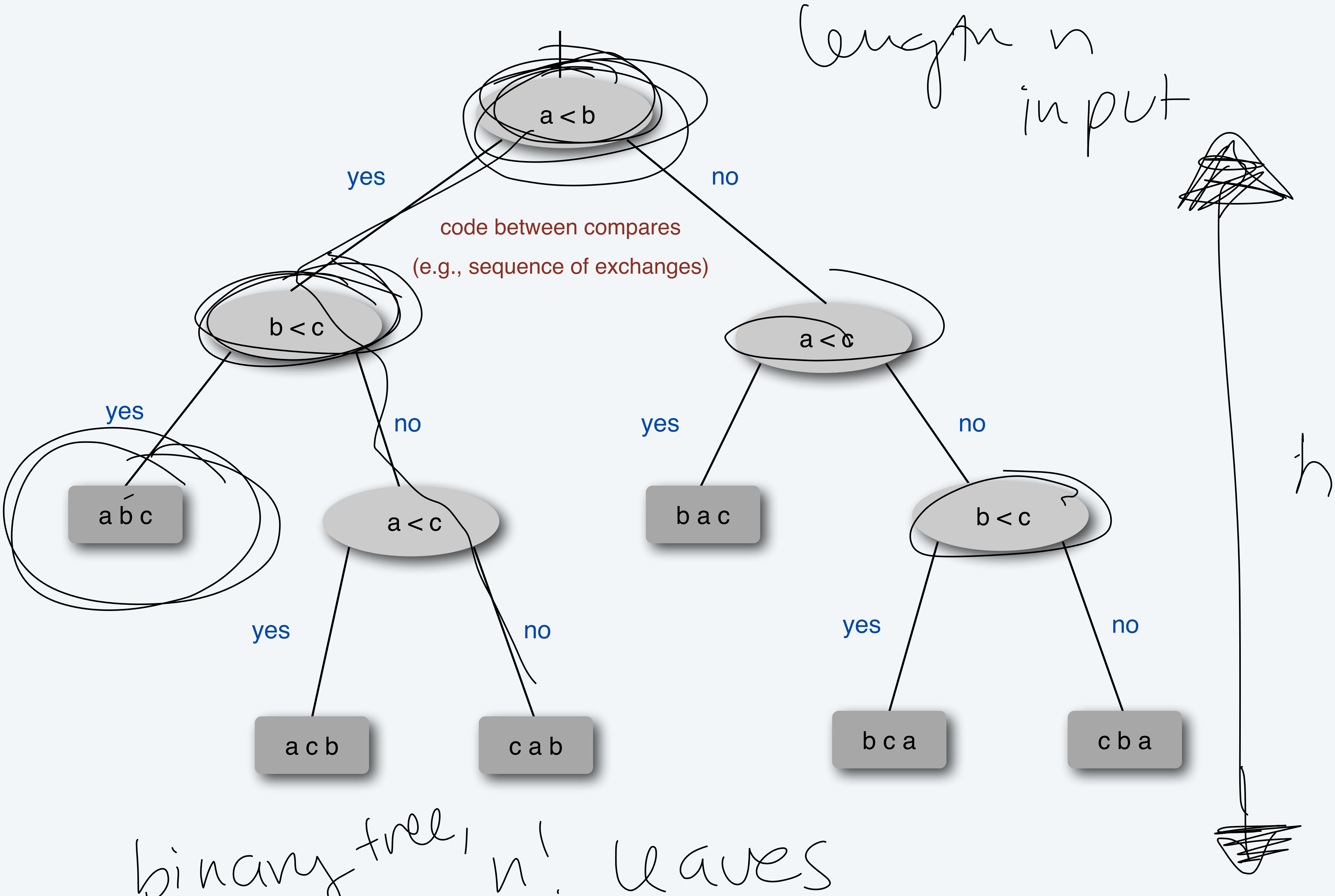

Sorting lower bound

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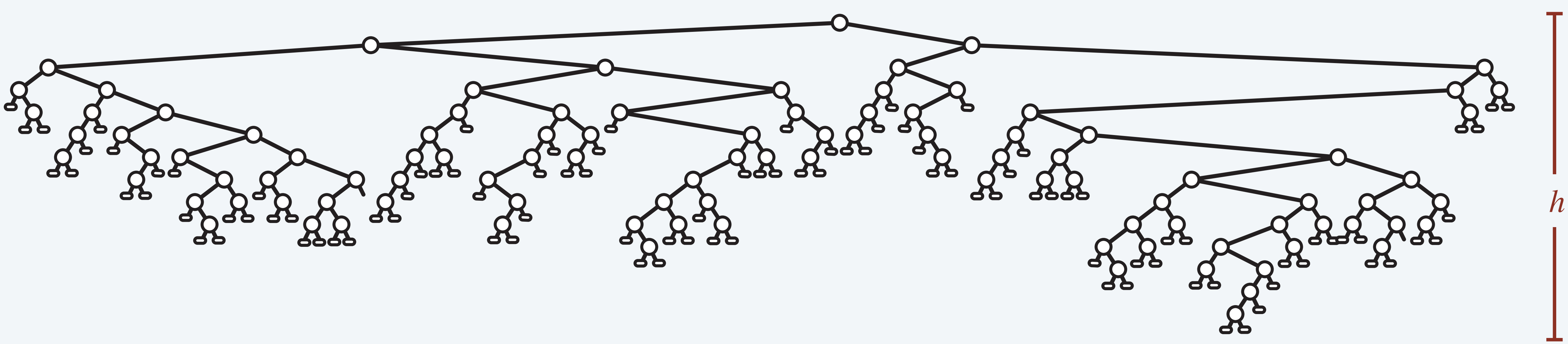

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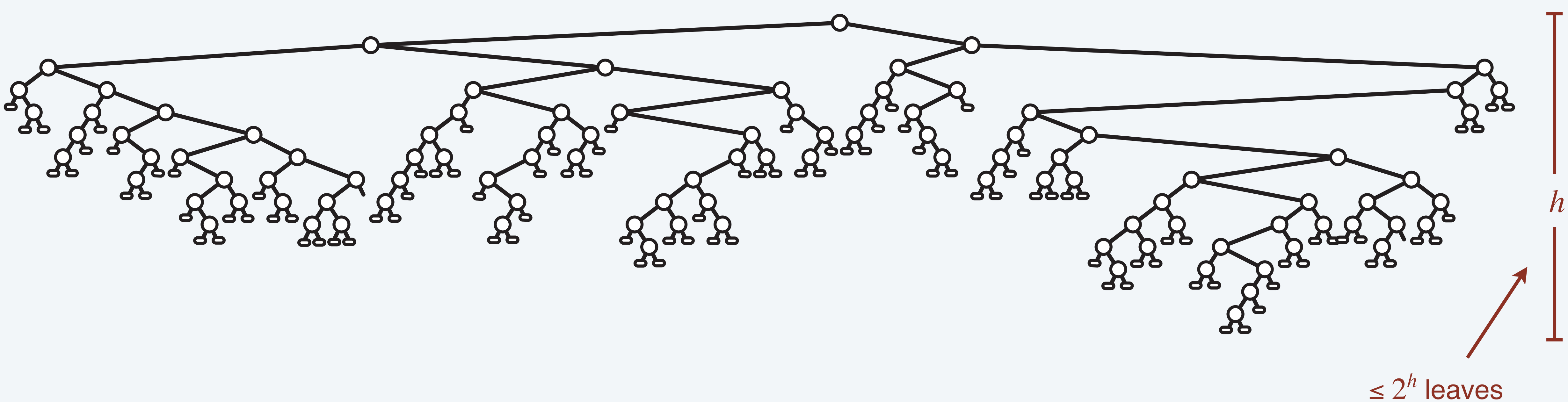

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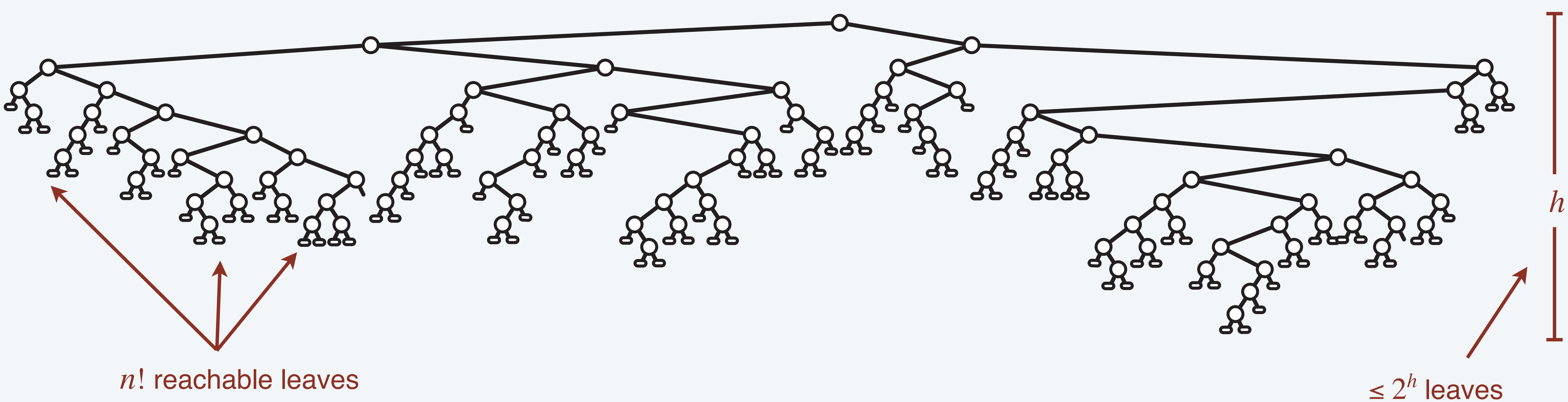

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Stirling's formula

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Note. Lower bound can be extended to include randomized algorithms.

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- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

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2 inversions: 3-2, 4-2