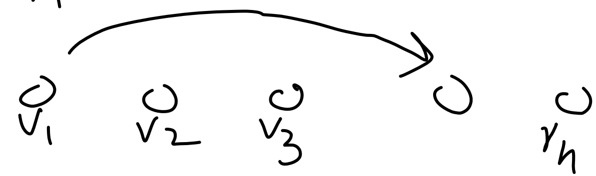
Last time, we:

Learned topological order of a directed graph.

Give some order $v_1, v_2, \dots, v_n$

For all edges (v_i, v_j) $i < j$.



If a directed graph has a topo. order, then it is a DAG. (contradiction)

If a graph is a DAG, then it has a topo. order.

For all DAGs G, G has a topological order.

If G is a DAG, then G has a node w/no entering edges. (contradiction)

Proof boilerplate: induction

Theorem: Every Y has quality Z.

Let x be an arbitrary Y.

Suppose that for all w less than (smaller than) x, quality Z holds. (inductive hypothesis, IH)

There are two cases:

Case 1: base case, we can prove directly that the theorem holds.

Case 2: inductive case. We need to use inductive hypothesis to show that theorem holds.

~~x~~ has quality Z. Because x was an arbitrary Y, every Y has quality Z.

G

Proof by induction that all DAGs have a topological ordering

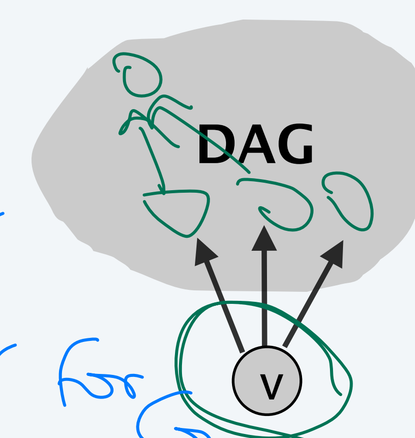
Let G be an arbitrary DAG.

Suppose that for all DAGs smaller than G have a topological order.

There are two cases:

Case 1: Base case. Suppose G has one node. G has a topo. order.

Case 2: Suppose G has two or more nodes. Notice that G must have a node w/no entering edges. Remove that node from G to form G'. By the IH, G' has a topo. order $v_1, v_2, \ldots, v_{n-1}$. Because v had no entering edges, $v, v_1, v_2, \ldots, v_{n-1}$ is a topo. order for G.

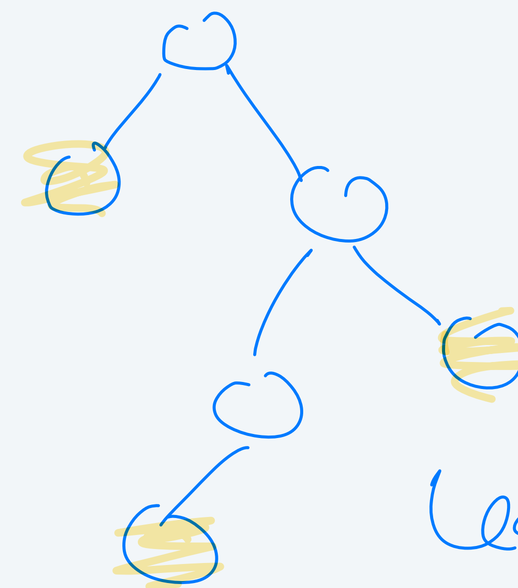


So G has a topo. ordering. Since G was arbitrary, every DAG has a topo. order.

Your turn

A binary tree is a rooted tree in which each node has at most two children. Show by induction that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.

DO NOT PROVE IT! Write the boilerplate.



leaves = nodes w/no children

Theorem: Every Y has quality Z.

Let x be an arbitrary Y.

Suppose that for all w less than x, quality Z holds.

There are (at least two) cases:

Case 1: non inductive case, aka base case. can prove directly that theorem holds. (but there could be more than one of these!)

Case 2: inductive case. need to use inductive hypothesis to show that theorem holds. (but there could be more than one of these!)

x has quality Z. Because x was an arbitrary Y, every Y has quality Z.

Your turn

integers

For all $n > 0$, $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

DO NOT PROVE IT! Write the boilerplate.

Every integer n greater than 0 has quality that $1+2+\dots+n = \dfrac{n(n+1)}{2}$

Theorem: Every Y has quality Z.

Let x be an arbitrary Y. Let n be an integer greater than 0.

IH Suppose that for all w less than x, quality Z holds. Suppose that for all $w < n$, $1+2+\cdots+w = \dfrac{w(w+1)}{2}$.

There are (at least two) cases:

Case 1: non inductive case, aka base case. can prove directly that theorem holds.

(but there could be more than one of these!) Base case: $n=1$.

Case 2: inductive case. need to use inductive hypothesis to show that theorem

holds. (but there could be more than one of these!) Inductive case: Assume $n>1$. Consider $n-1$.

x has quality Z. Because x was an arbitrary Y, every Y has quality Z.

$1+2+\dots+n = \dfrac{n(n+1)}{2}$. Because n was arbitrary, the theorem holds.