Plan for today

Quiz

Go over quiz

Last challenge problem: segmented least squares

1. (3 points) Suppose you get input prices for days 1 through 10 as

$$797, 309, 850, 472, 398, 419, 383, 955, 279, 619$$

.

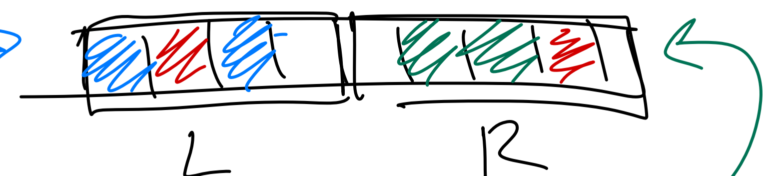
What is the maximum profit for this input?

646

2. (3 points) We would like to write a divide and conquer algorithm for this problem. Just as we have done for many other problems, we will do this by dividing the array in half, using recursion to find the optimal solution in each half, and then finding a way to merge the optimal solutions from each half into one single optimal solution.

Let the array you are trying find the max profit for be A. If you divide array in A into two halves, L and R (for left and right), the max profit is made in exactly one of these ways:

- Buying and selling on days in L
- Buying and selling on days in R
- ??? Buy in L, sell in R



What is the third way that you could make the max profit from the prices in A?

3. (3 points) What is a base case for the max profit problem? You may assume that the number of days is a power of 2 if you would like.

n=1 : return 0

[100]

---

n=2 return max(price day$_2$ - prize day$_1$, 0)

1 2

[300, 250]

Max_Profit($p_1, p_2, \ldots, p_n$):

if $n = 2$:

  return max(0, $p_2 - p_1$)

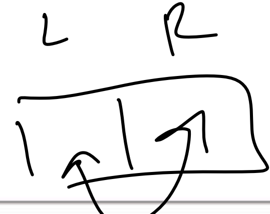
else:

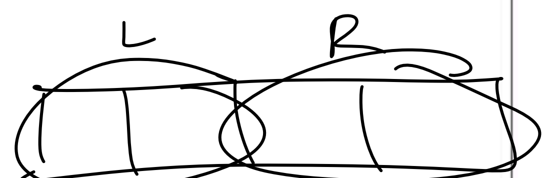
  m_L = Max_Profit(L half of p_1 to p_n)

  m_R = Max_Profit(R half of p_1 to p_n)

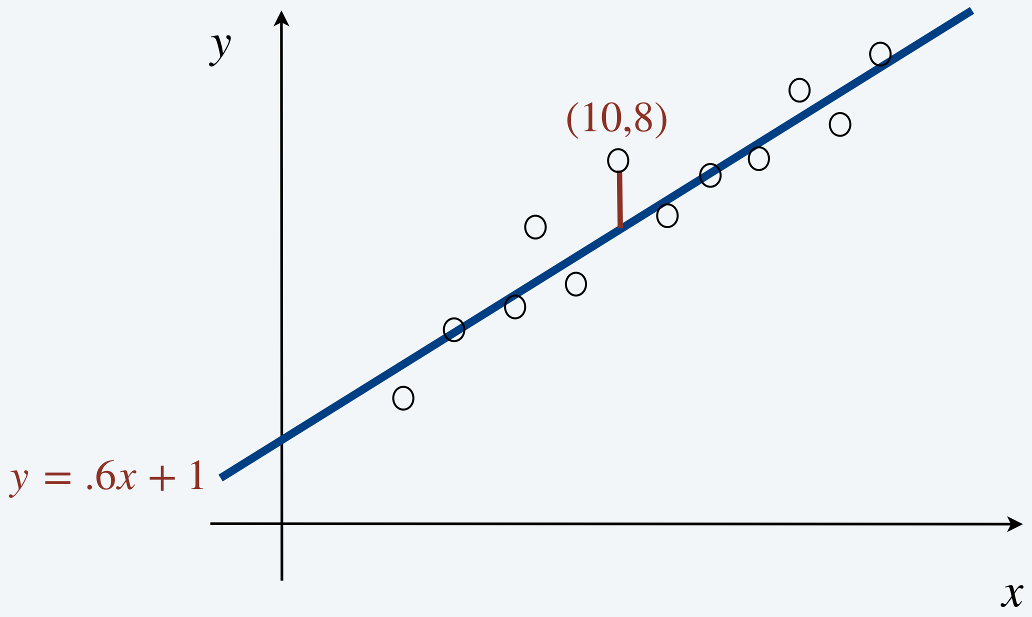
  m_X = max(R) − min(L)

  return max(m_L, m_R, m_X)



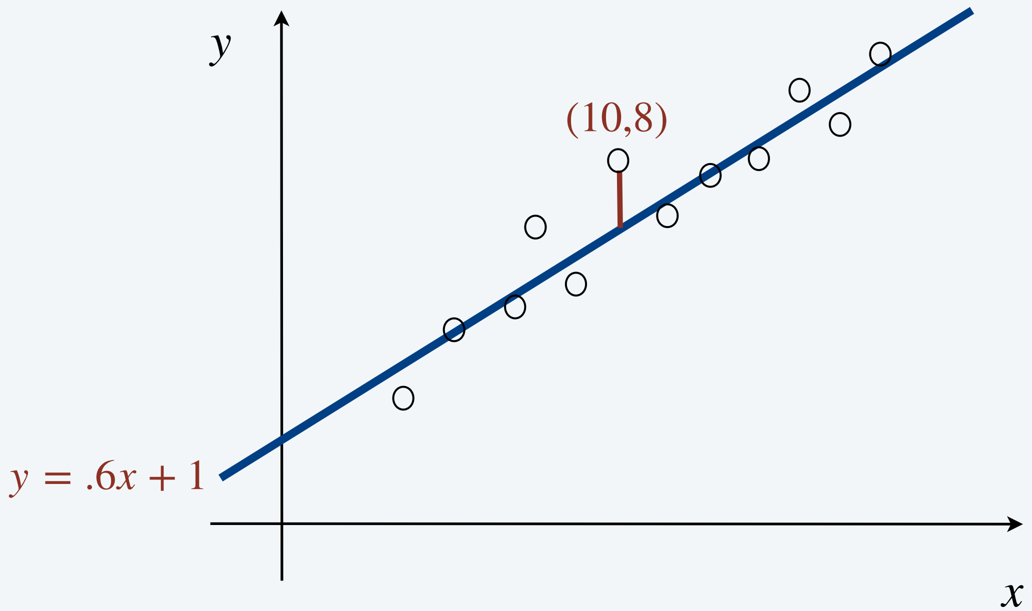


Least squares



Least squares

Least squares. Foundational problem in statistics.



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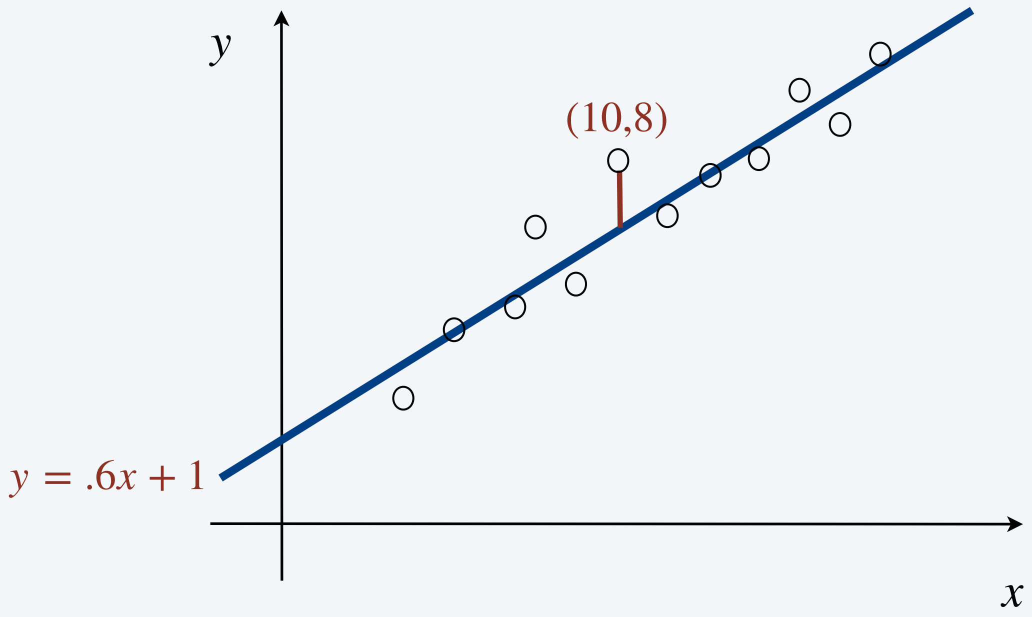

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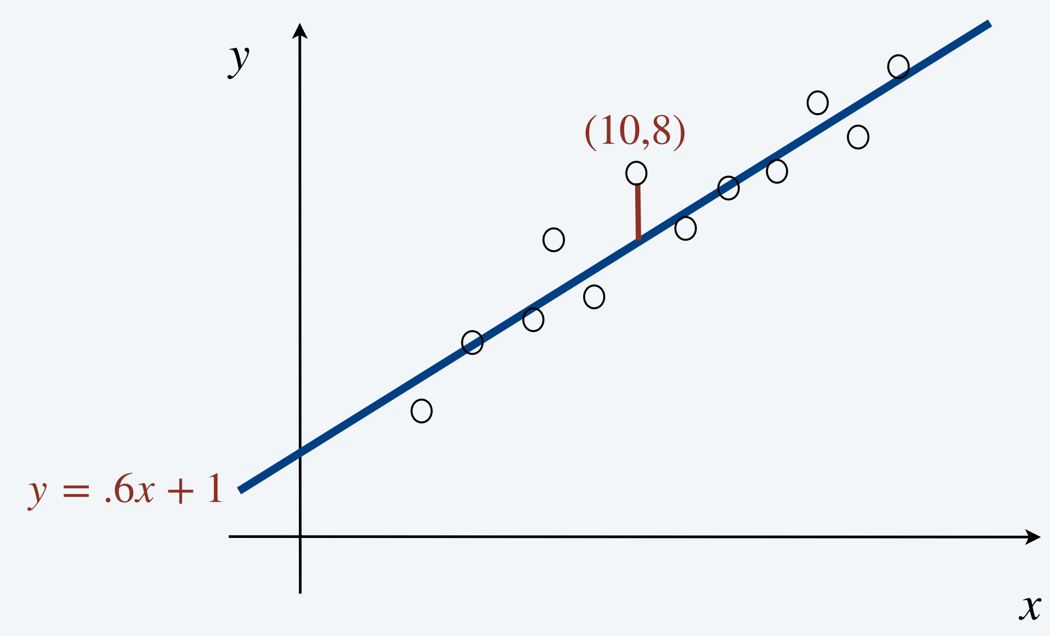

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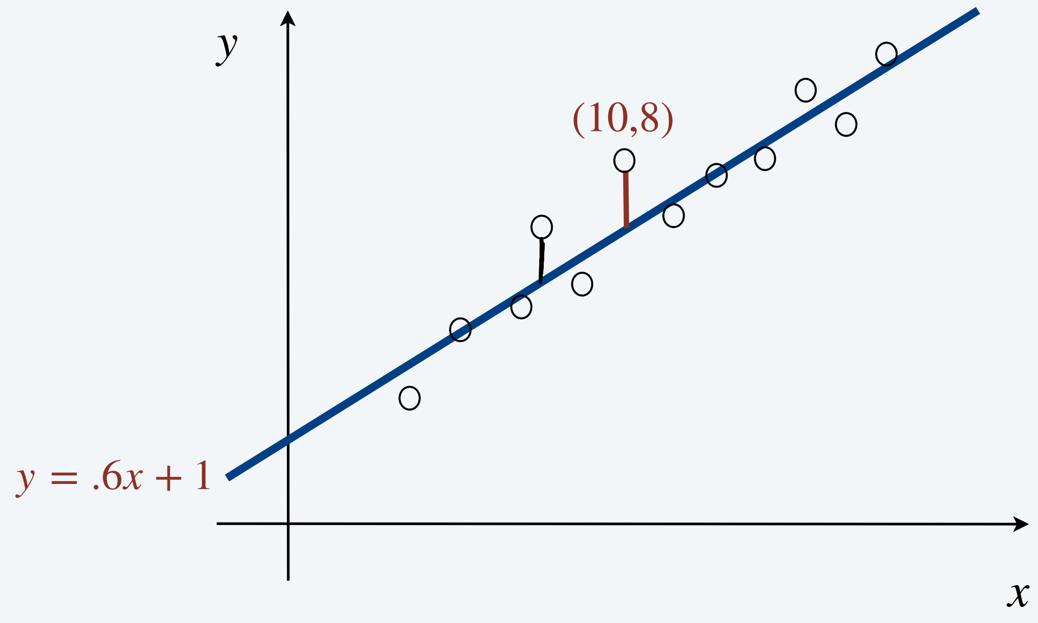

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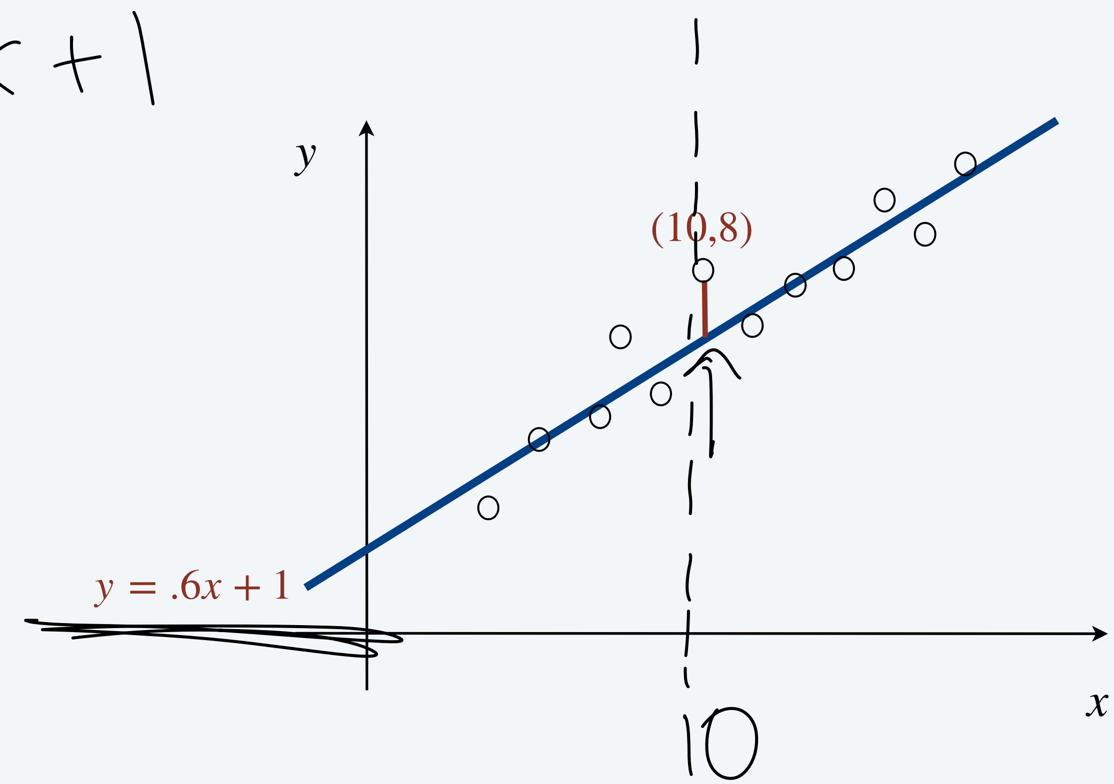
(10, 8)
x coord y coord

$y = .6x + 1$

$(8 - 7)^2 = 1^2 = 1$



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given point y val

y val of line

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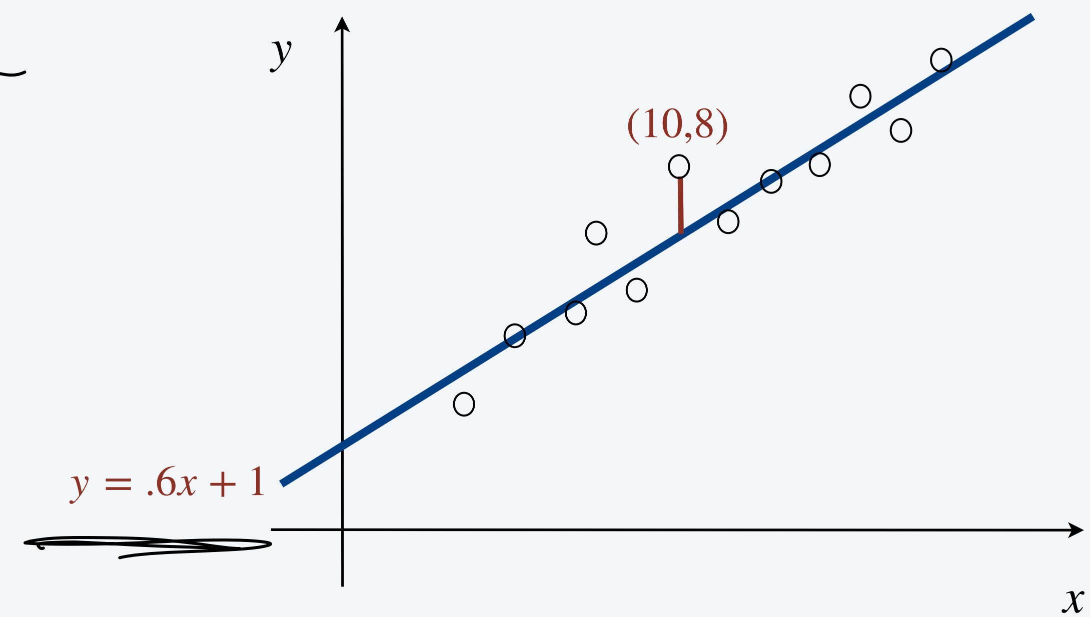

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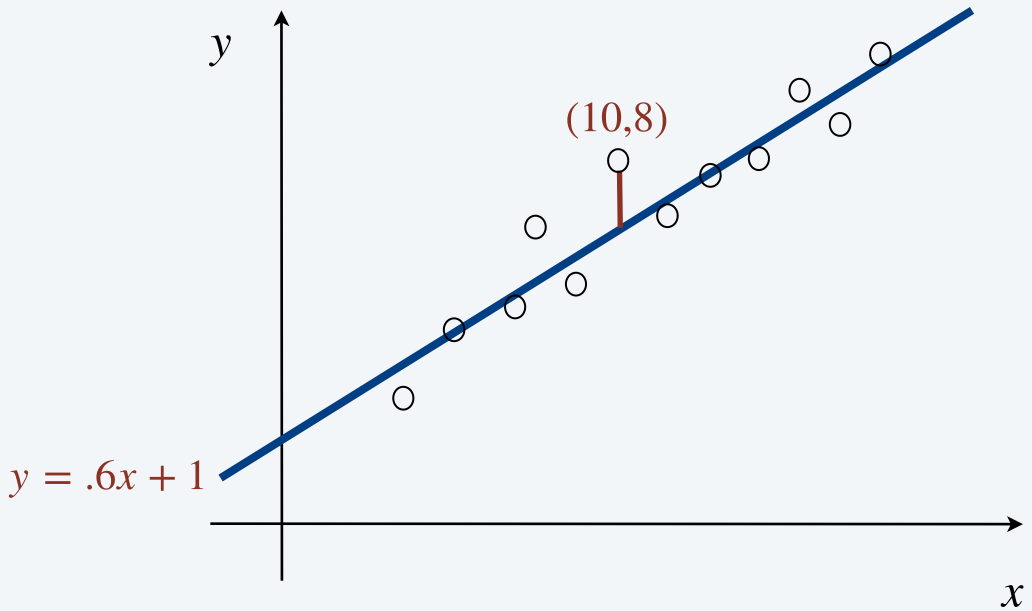

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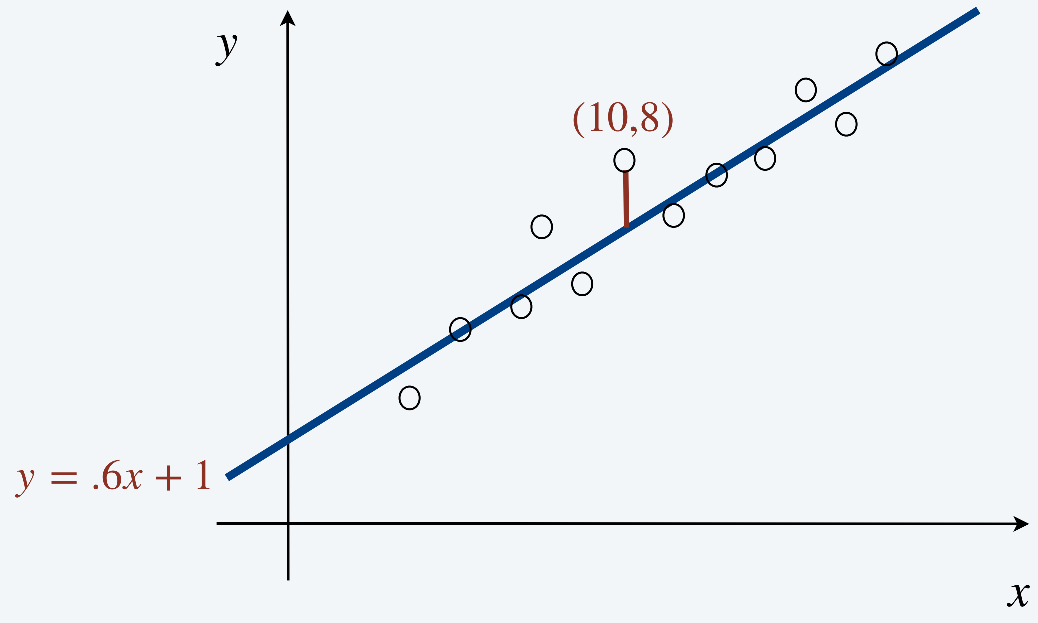

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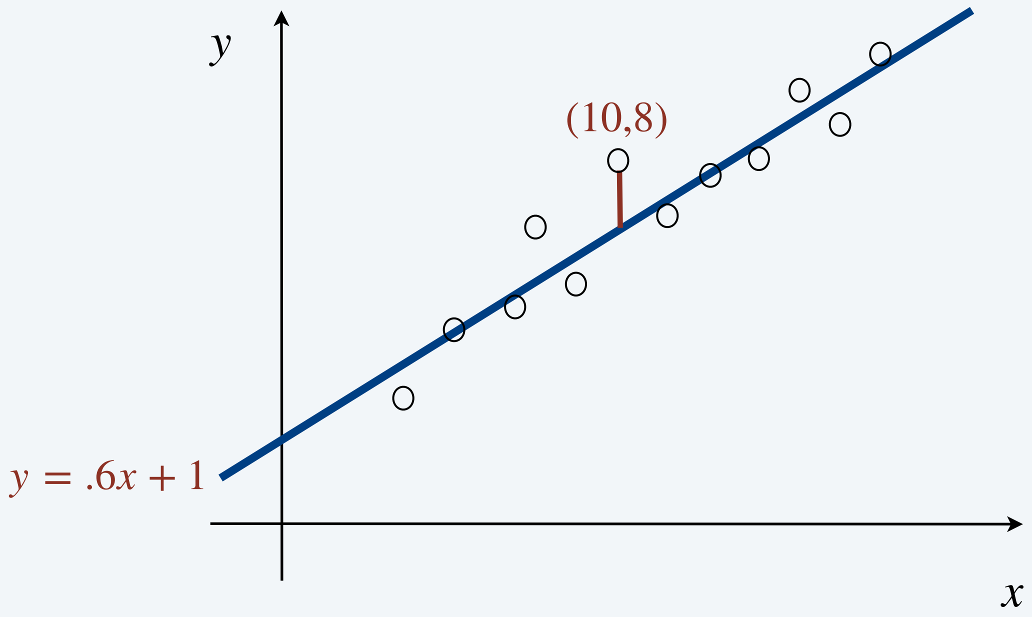

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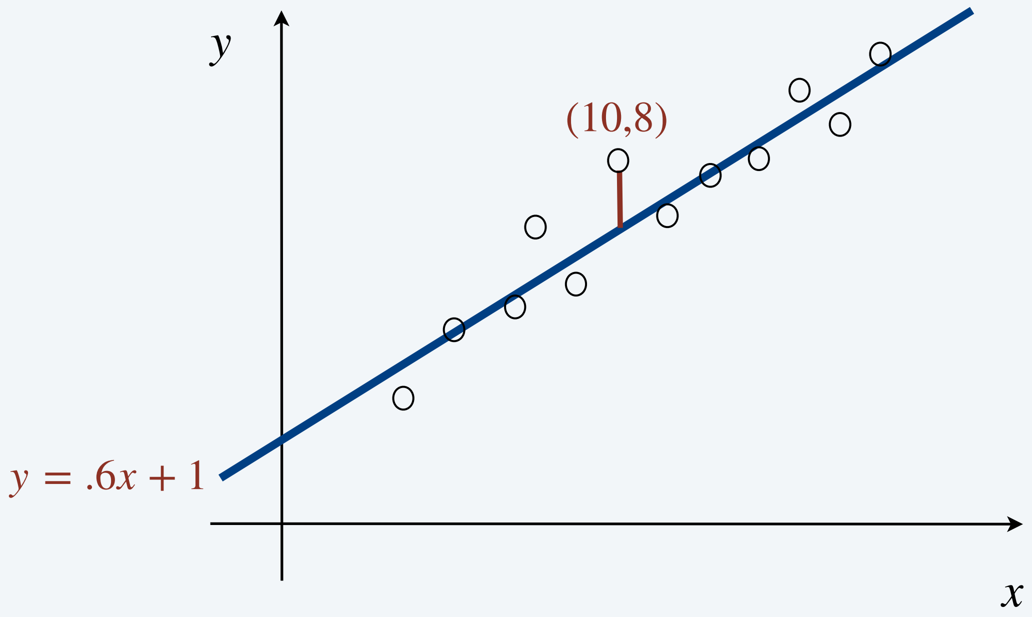

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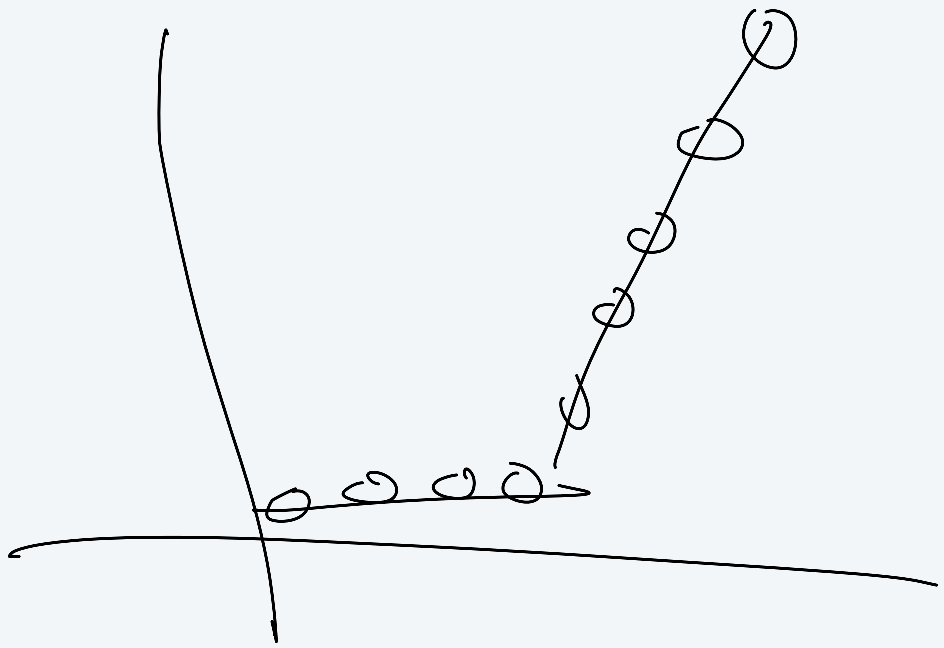
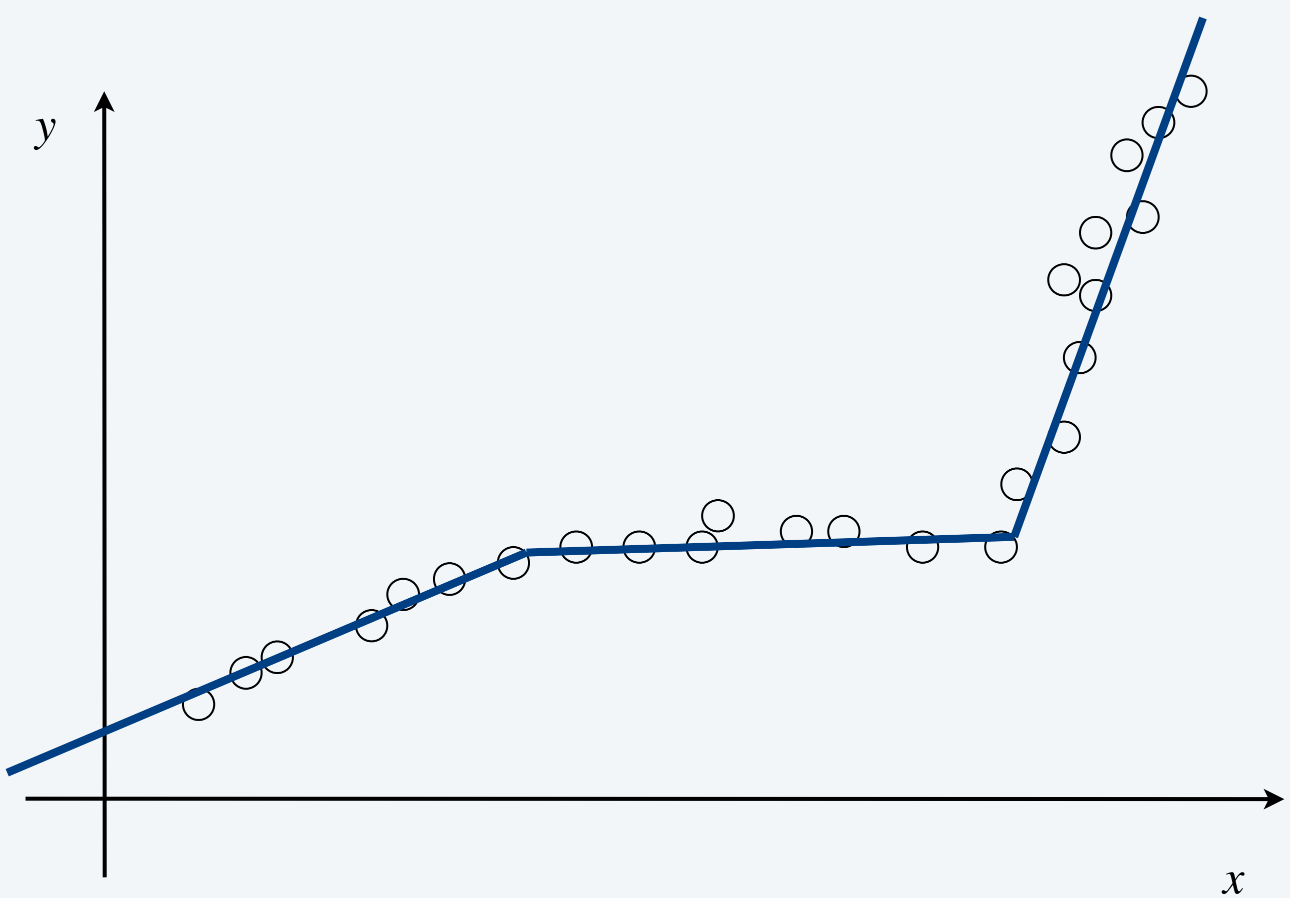
Segmented least squares





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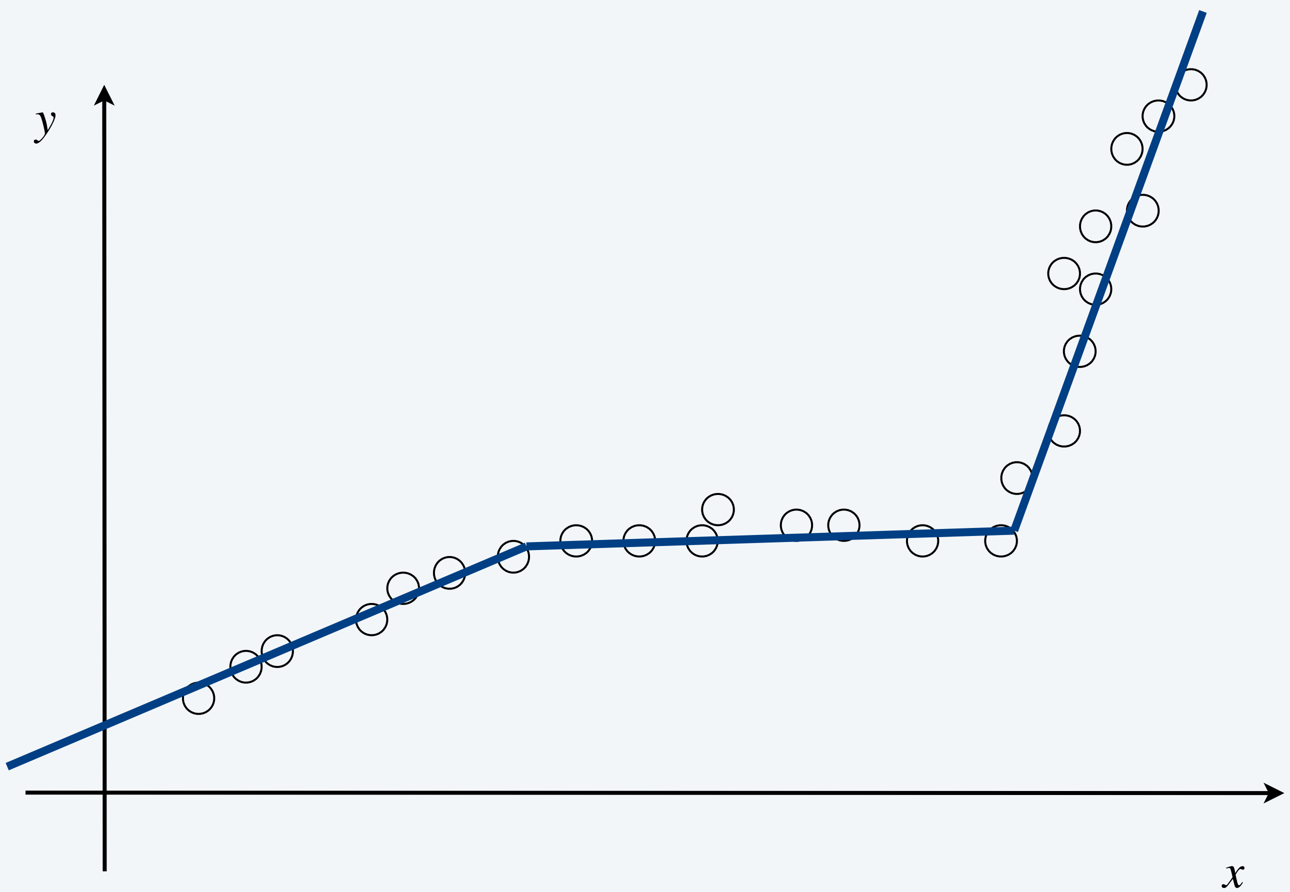

Segmented least squares

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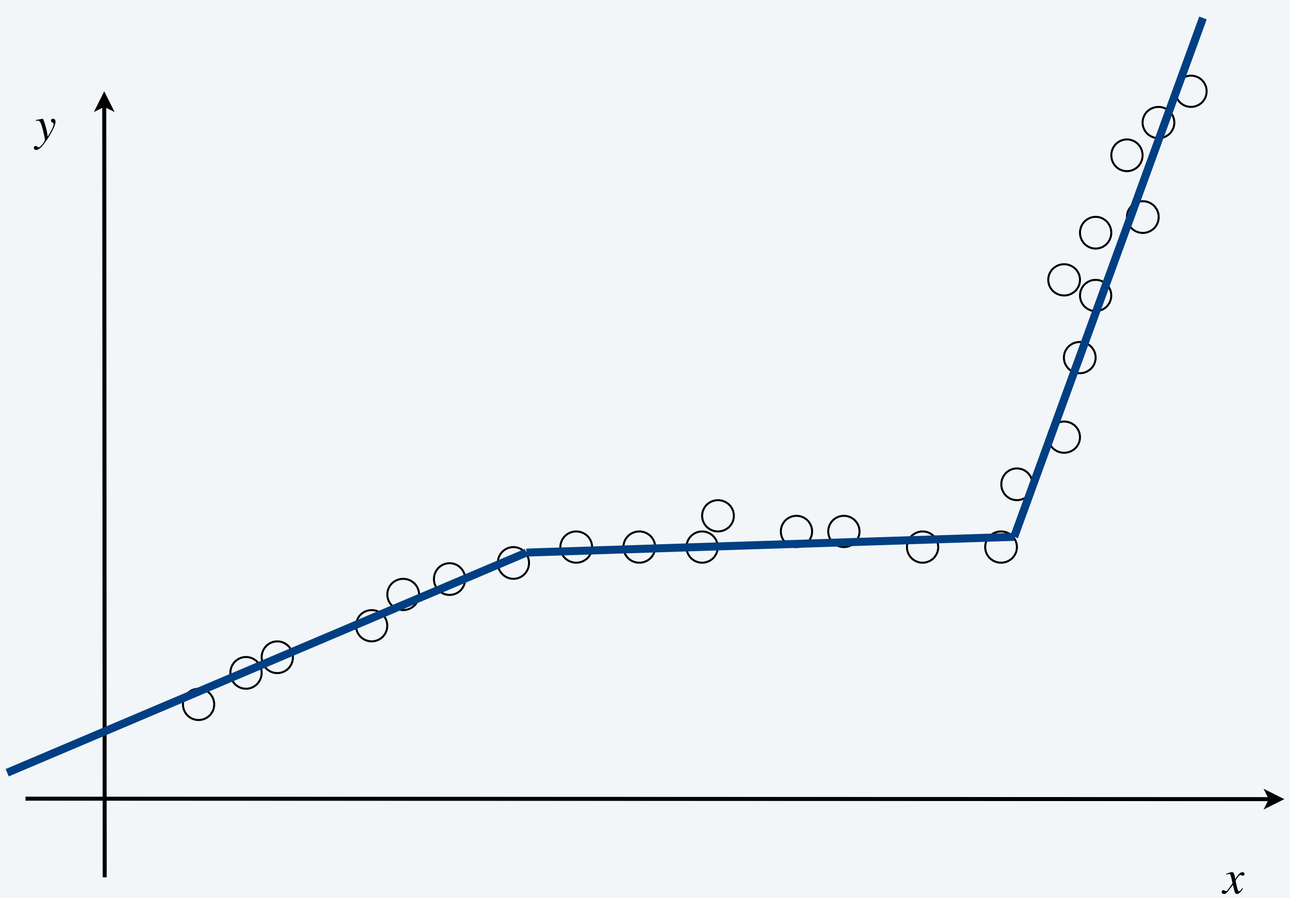
- Points lie roughly on a sequence of several line segments.



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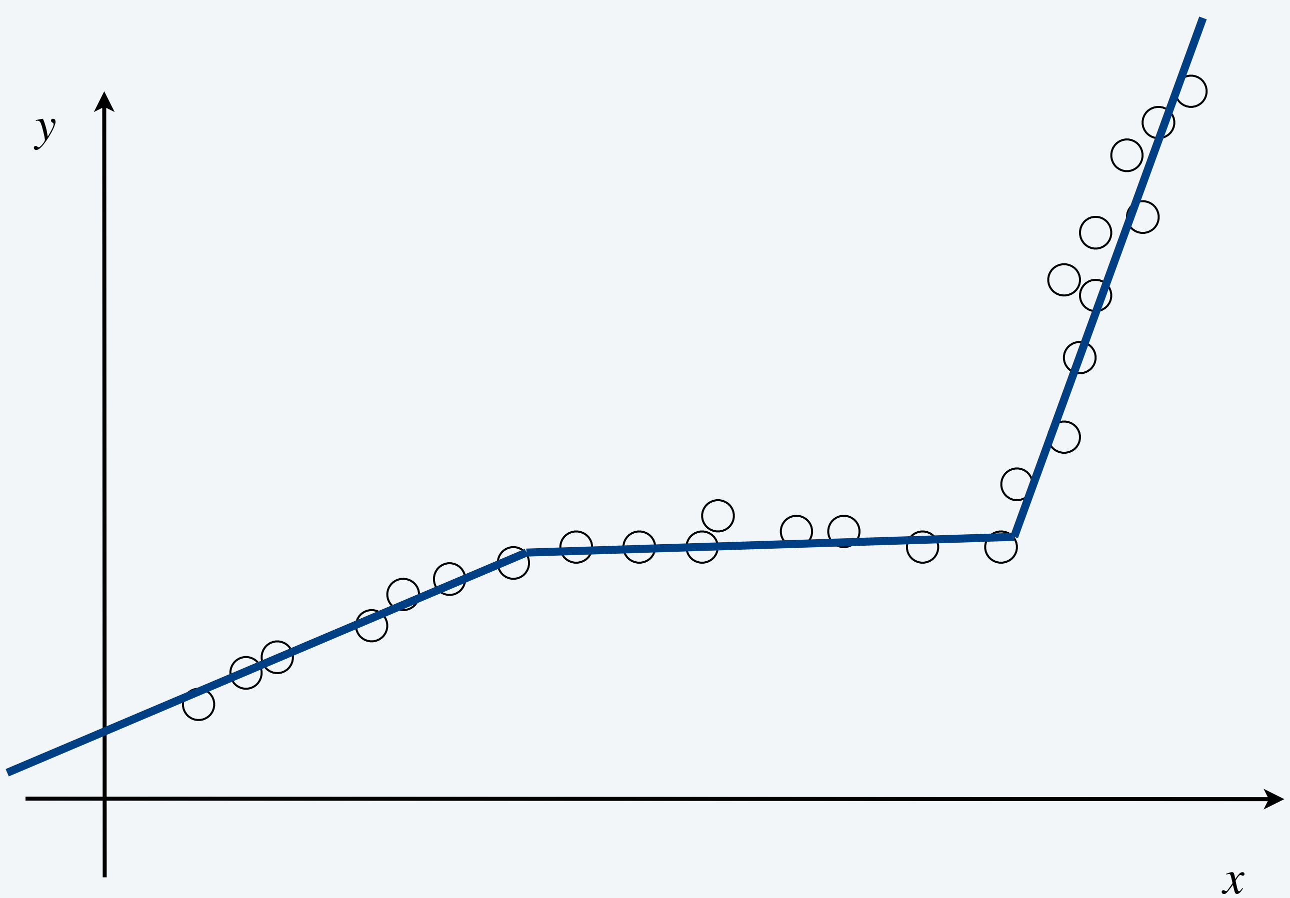

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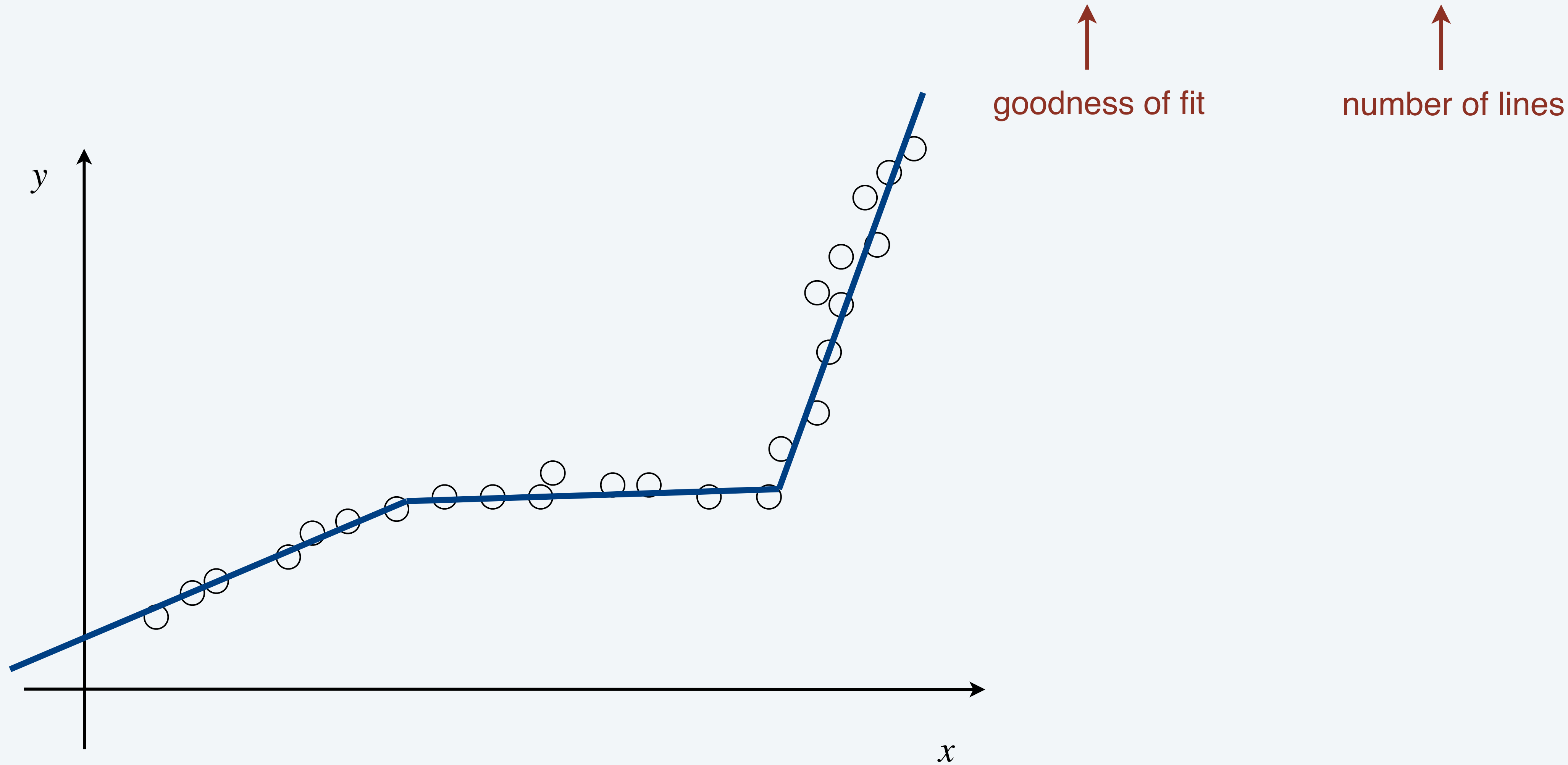
Q. What is a reasonable choice for $f(x)$ to balance accuracy and parsimony?



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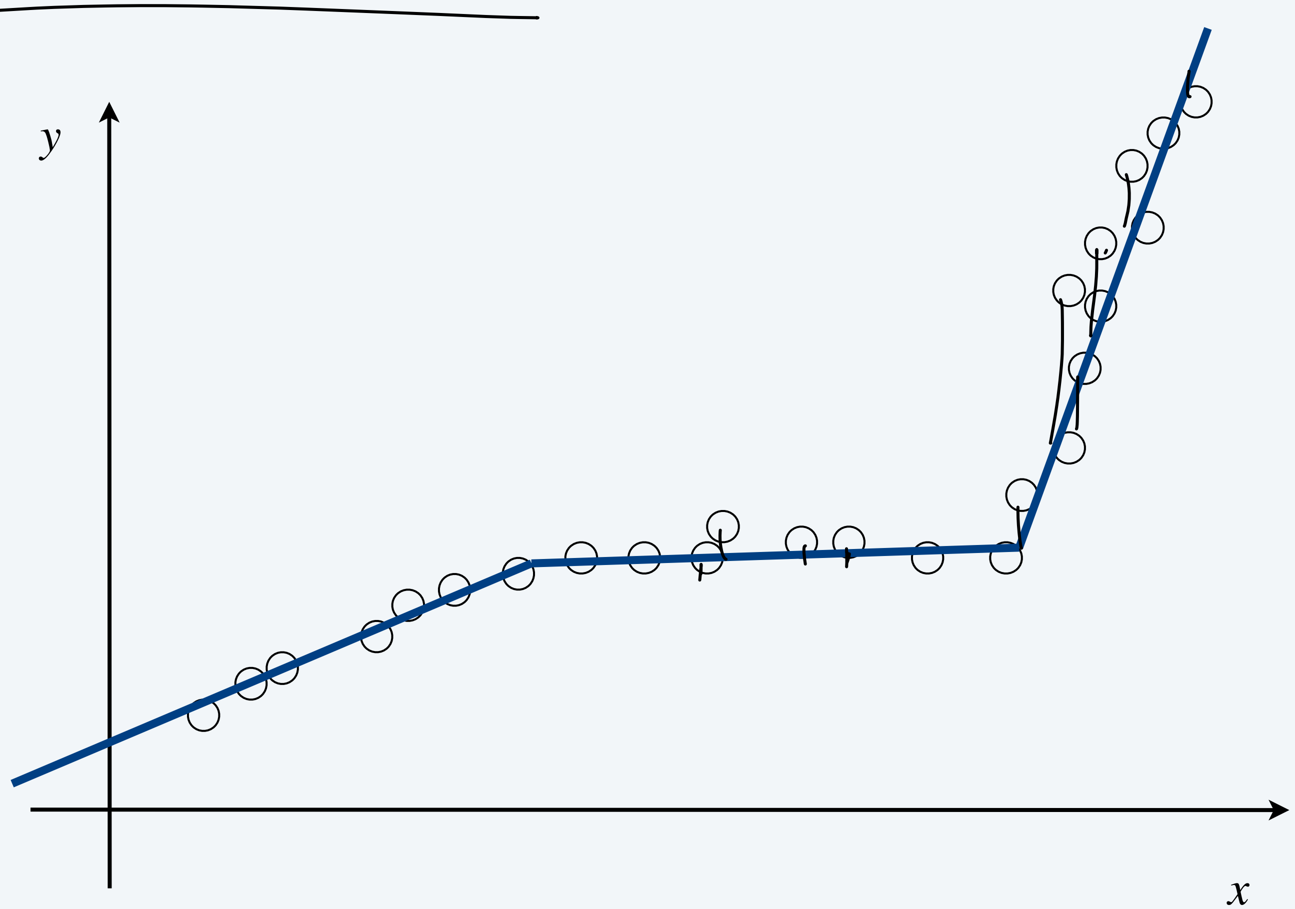
Goal. Minimize $f(x) = E + c\,L$ for some constant $c > 0$, where

- E = sum of the sums of the squared errors in each segment.
- L = number of lines.



$L = 3$

c

E

Dynamic programming: multiway choice

$(x_1, y_1), (x_2, y_2), (x_3, y_3) \dots (x_n, y_n)$

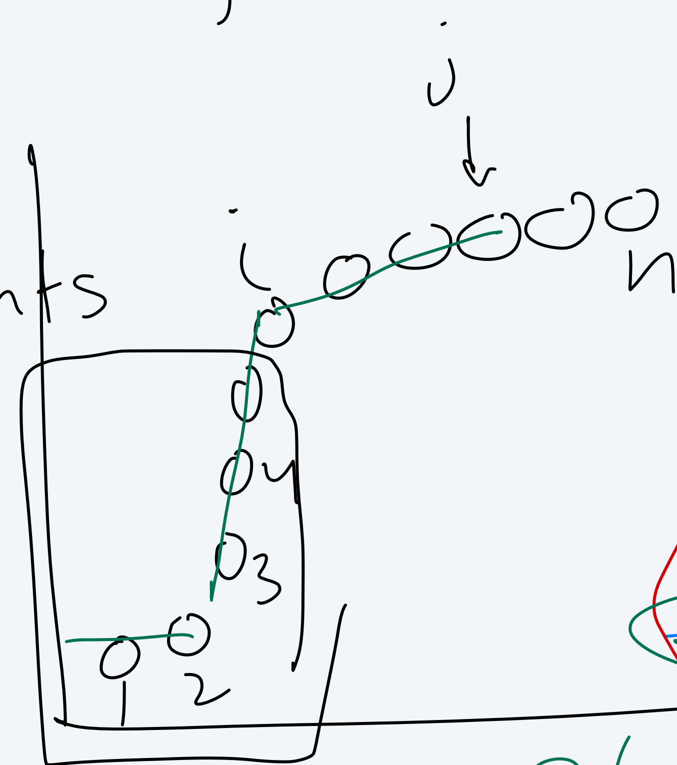
$OPT(j)$ = minimum value of $f(x)$ using all points up to point j

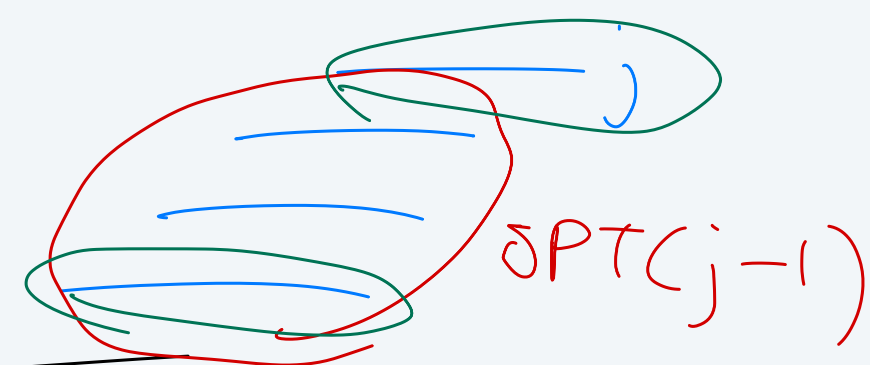
$f(x) = E + cL$

$OPT(j) = c + e_{ij} + OPT(i-1)$

- c: cost of line
- e_{ij}: cost of errors on line
- points i through j are on a line

cost of line through points i to j





$opt(p(j)) + w_j$

Dynamic programming: multiway choice

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- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some $i \leq j$.

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- Cost = $e_{ij} + c + OPT(i-1)$.

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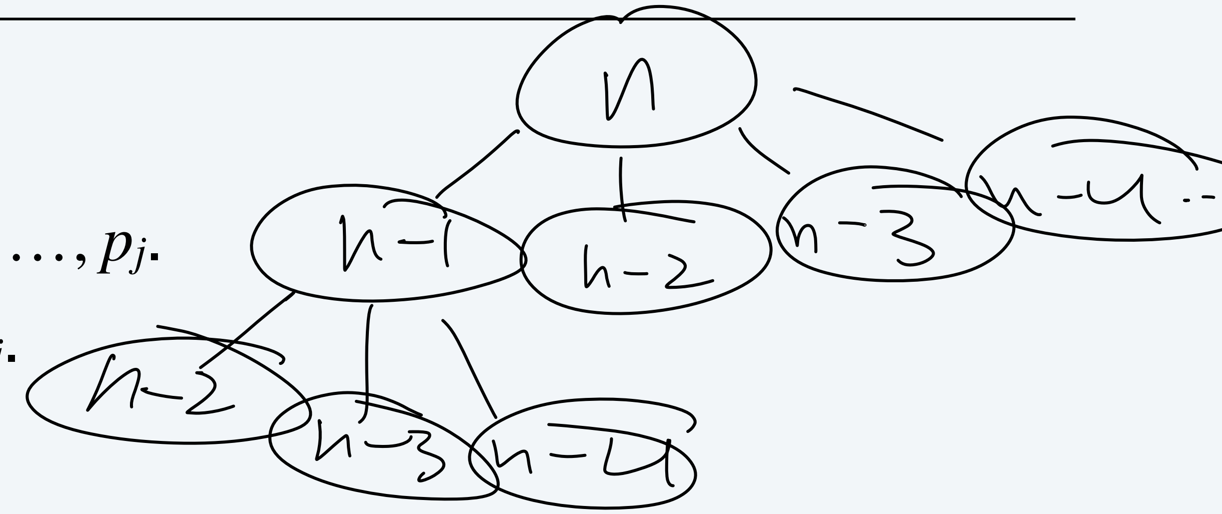

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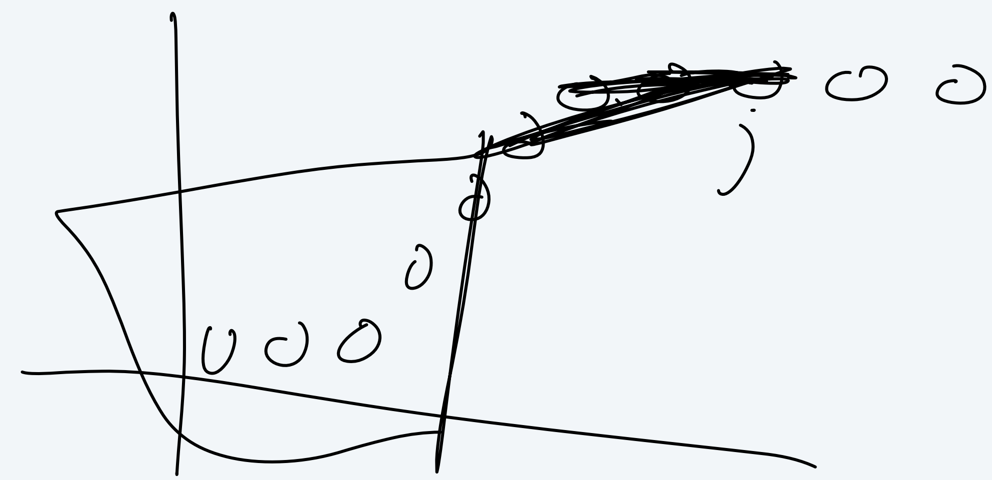

Answer: consider all of them!

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ e_{ij} + c + OPT(i-1) \} & \text{if } j > 0 \end{cases}$$

Segmented least squares algorithm

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SEGMENTED-LEAST-SQUARES($n, p_1, \ldots, p_n, c$)

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SEGMENTED-LEAST-SQUARES(n, p_1, …, p_n, c)

FOR j = 1 TO n

  FOR i = 1 TO j

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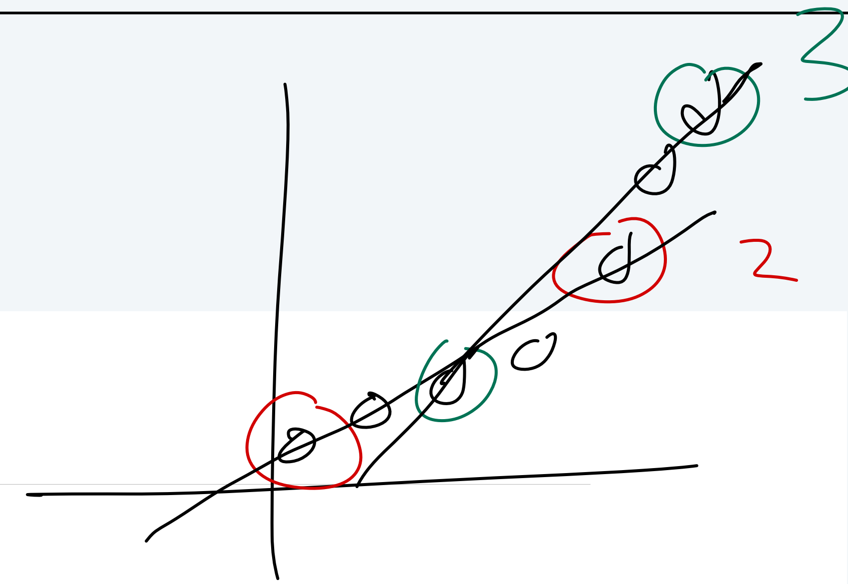

SEGMENTED-LEAST-SQUARES$(n, p_1, \ldots, p_n, c)$

pre-compute e_{ij} values

FOR j = 1 TO n

  FOR i = 1 TO j

    Compute the SSE e_{ij} for the points $p_i, p_{i+1}, \ldots, p_j$.

$\Theta(n^3)$

runtime?

what's a lower bound?

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runtime?

$M[0] \leftarrow 0$.

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previously computed value

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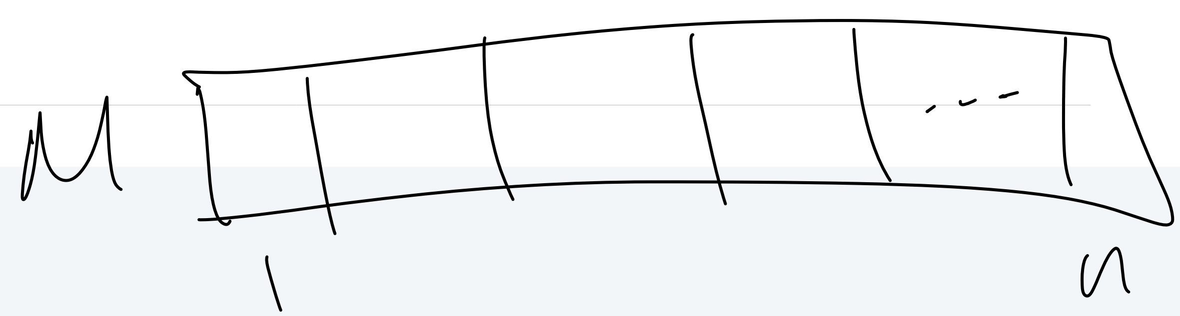

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Theorem. [Bellman 1961] DP algorithm solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.

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- Bottleneck = computing SSE e_{ij} for each i and j.

$$a_{ij} \;=\; \frac{n \sum_k x_k y_k - (\sum_k x_k)(\sum_k y_k)}{n \sum_k x_k^2 - (\sum_k x_k)^2}, \quad b_{ij} \;=\; \frac{\sum_k y_k - a_{ij} \sum_k x_k}{n}$$

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- Using cumulative sums, can compute e_{ij} in $O(1)$ time.