Today: exams back this afternoon!

* any questions on exam 1?

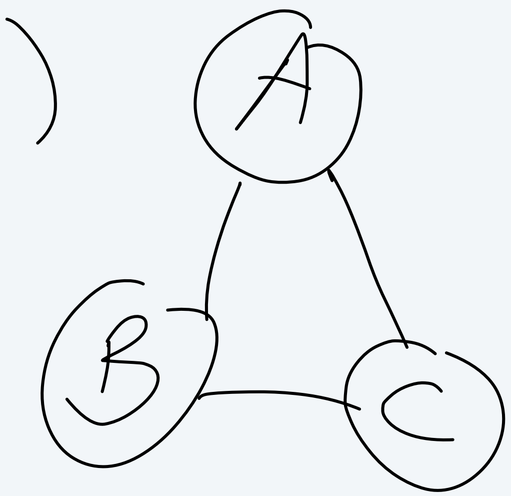
* learn topological ordering definition

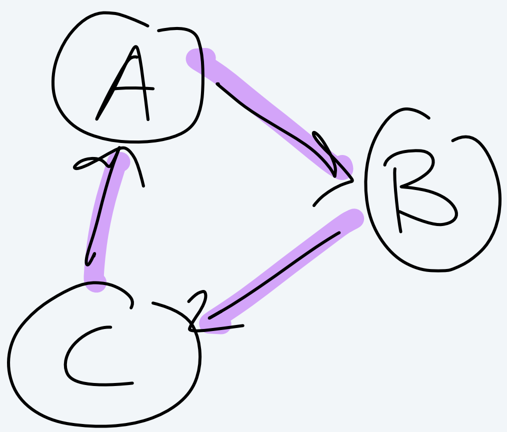
* see a proof by induction about graphs (you have another one on your homework!)

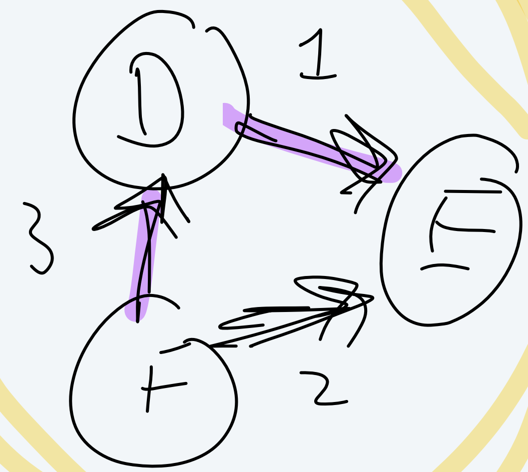
depending on how much time we spend on exam 1, break may be earlier or later

Directed acyclic graphs

Def a directed acyclic graph (DAG) is a directed graph that contains no directed cycles.







DAG (has an undirected cycle)

F, D, E

Def A topological order of a directed graph is an ordering of the nodes as $v_1, v_2, \dots, v_n$ such that for every edge (v_i, v_j) we have $i < j$.

order 1   D, E, F
v_1 v_2 v_3

$\overset{1}{D} \to \overset{2}{E}$

$3\ F \to E\ 2$

$3\ F \to D\ 1$

With your table...

Def. A DAG is a directed graph that contains no directed cycles.

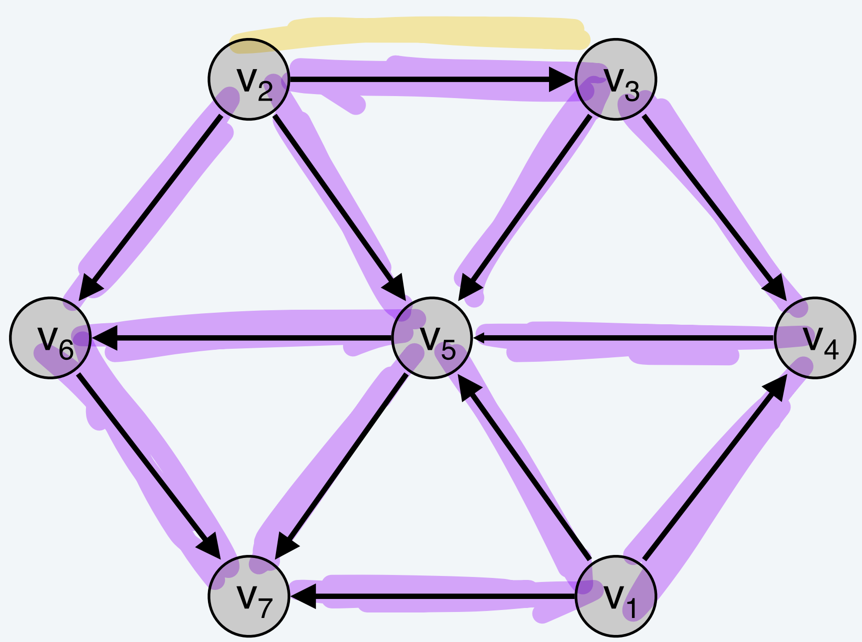
Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \dots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

v_i to v_j

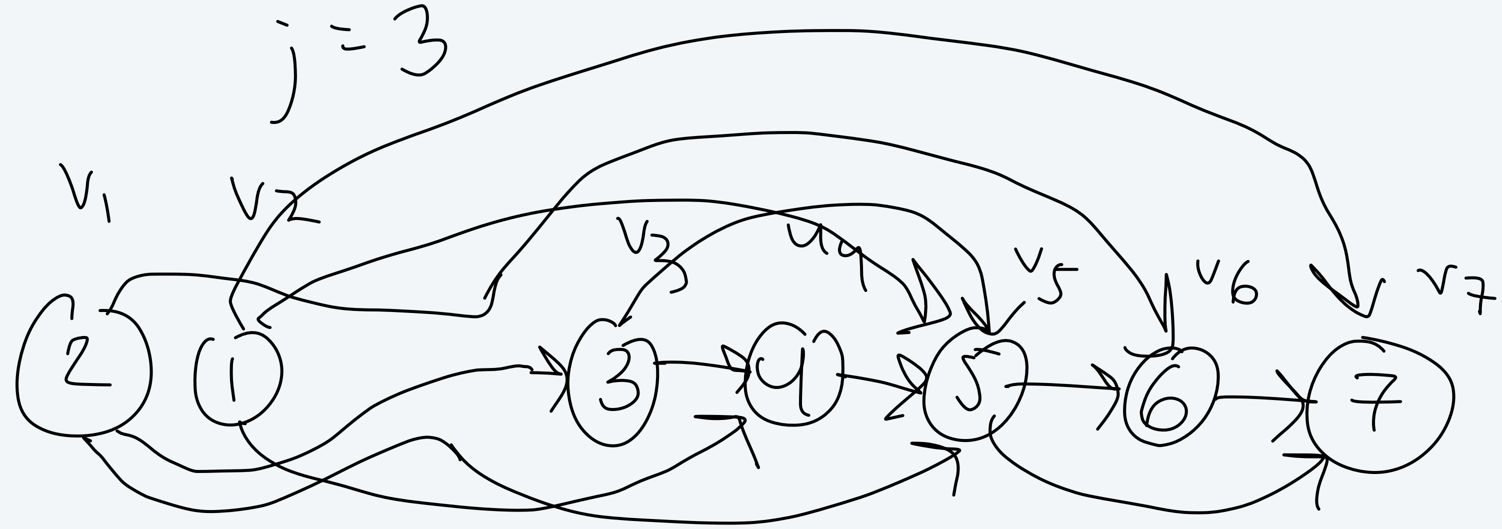
for edge v_2 to v_3

$i = 1$

$j = 3$



a DAG



edge v_1 to v_4 (v_i, v_j)

$i = 1$

$j = 4$

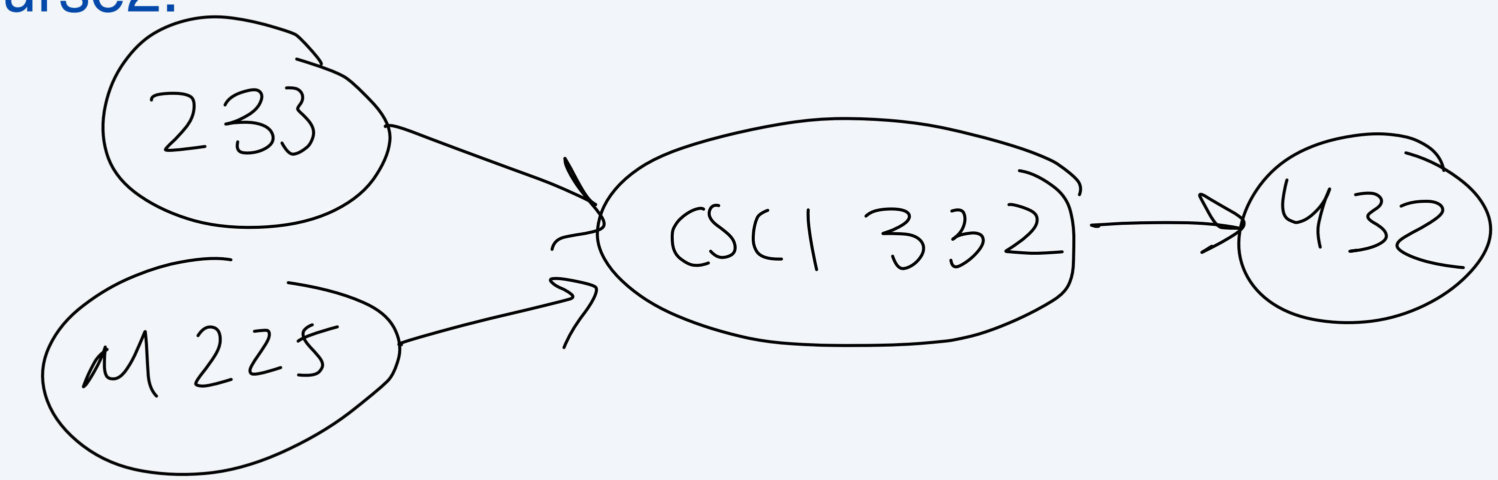
$1 < 4$ ✓

give a topological ordering

With your table...

Suppose I have a graph where nodes are courses that are required for a Computer Science degree and an edge between two nodes course1 and course2 indicates that course1 is a prerequisite for course2.

Consider the node for CSCI 332.



1) What edges enter it?

2) What edges leave it?

whole the graph of all classes

What does a topological order of this graph represent?

Is there more than one topological ordering?

break?

T or F: if G has a topological order, then G is a DAG. bad

Directed acyclic graphs

back at 10:22

If G has a topological order, then G is a DAG. → no cycles.

~~Q. Does every DAG have a topological ordering?~~

note: G must be directed.

Proof boilerplate: direct

Theorem: Every Y has quality Z. has no cycles.

directed graph with a topological ordering

Let x be an arbitrary Y.

let x be an arbitrary directed graph with a topological order.

has no cycles.

x has quality Z. Because x was an arbitrary Y, every Y has quality Z.

Proof boilerplate: contradiction

Theorem: Every Y has quality Z.

For the sake of contradiction, suppose that not every Y has quality Z.

do stuff

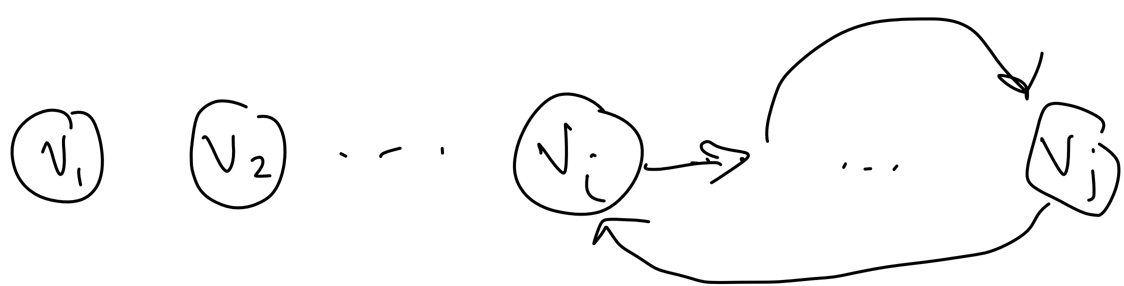
…a contradiction.

Thus, the assumption that not every Y has quality Z is incorrect, so the theorem is true.

Theorem: If G has a topological order, then G is a DAG.

For the sake of contradiction, suppose that not every directed graph with a topological order has no cycles. That is, suppose there is a graph G with topological order $v_1, v_2, \ldots, v_n$ and a directed cycle C.

Let v_i be the lowest-indexed node in C and let v_j be the node just before v_i.



But now there is an edge from v_j to v_i, with $i < j$, a contradiction.

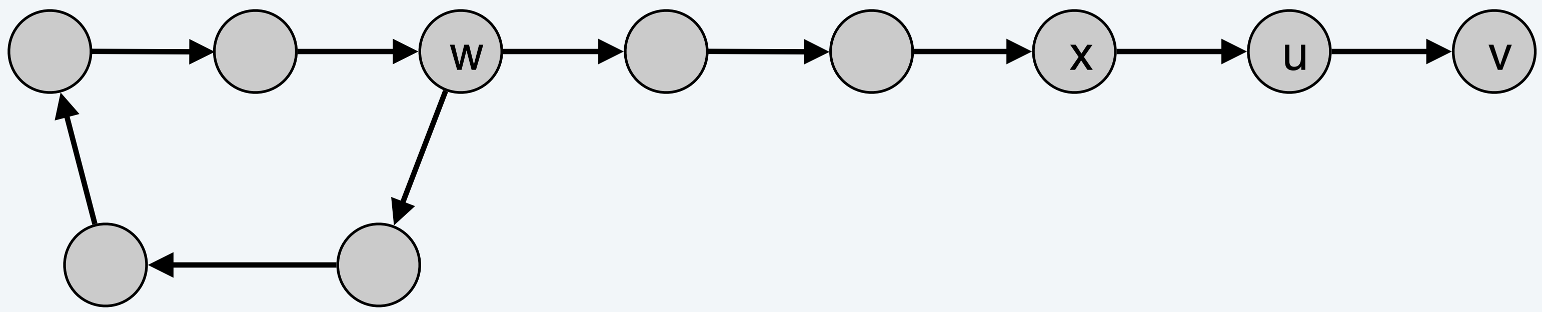
So our assumption that not every directed graph with a topo. order is acyclic is False.

Directed acyclic graphs

Lemma. If G is a DAG, then G has a node with no entering edges.

Pf. [by contradiction]

- Suppose that G is a DAG and every node has at least one entering edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one entering edge (u, v) we can walk backward to u.
- Then, since u has at least one entering edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪



Lemma. If G is a DAG, then G has a topological ordering.

Proof boilerplate

If G is a DAG, then G has
a topo. order.

Theorem: Every Y has quality Z.

DAG

has a topo. order.

Let x be an arbitrary Y.

DAG.

do stuff → next time: fill in
w/ a proof by induction

DAG DAG a topo. order.

x has quality Z. Because x was an arbitrary ~~Y~~, every ~~Y~~ has ~~quality Z~~.

a topo. ordering

Theorem If G is a DAG, then G has a node with no entering edges.

Proof

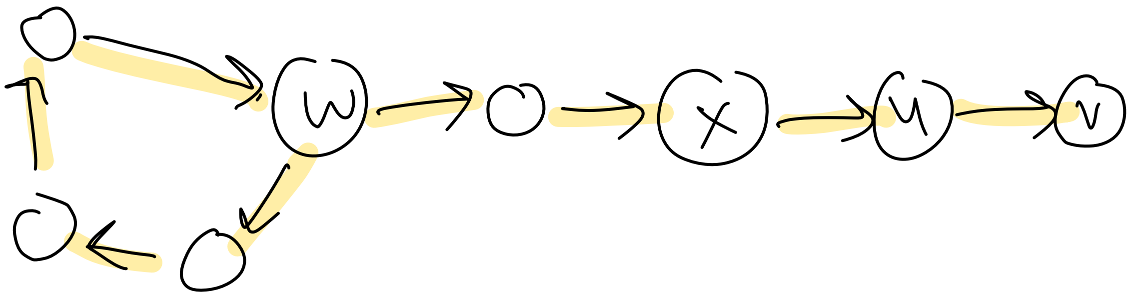
we prove by contradiction.

Suppose that it is not true that if G is a DAG, then it has a node with no entering edges.

That is, suppose there is a graph G that is a DAG and has no node with no entering edges.

Pick any node v and begin following

edges backward from v.
Continue until we visit a node twice.



But this is a contradiction.
So the initial assumption must be false.