Name _____

CSCI 332, Fall 2025

Exam 1 Practice 2

Note that this exam has four sections. They are:

1. Stable Matching (30 points)
2. Algorithm Analysis (30 points)
3. Graph Algorithms (30 points)
4. Greedy Graph Algorithms (10 points)

You may use a double sided 3x5 handwritten notecard of notes during the test but no other resources. If you need more space than what is given, develop your solution on scratch paper before copying your final answer to the exam paper.

Good luck!

Section 1 (Stable Matching)

Recall the Gale-Shapley Algorithm for solving the stable matching problem:

```
Gale-Shapley(preference lists for n hospitals and n students)
  Let M be an empty matching
  While some hospital h is umatched and hasn't proposed to every student:
      Let s be the first student on h's list to whom h has not yet proposed
      If s is unmatched:
          Add (h,s) to M
      Else s is matched to h':
          If s prefers h to h':
              Remove (h',s) from M
              Add (h,s) to M
          Else s prefers h' to h:
              Do nothing
  Return M
```

1. (10 points) For each of the following claims about the Gale-Shapley algorithm on any input, mark whether they are true or false.

   (a) Once a hospital becomes matched, it never becomes unmatched. T or F

   (b) Once a student becomes matched, it never becomes unmatched. T or F

2. (5 points) Describe a best-case input for the Gale-Shapley algorithm.

3. (10 points) Give the smallest, simplest function $f(n)$ that you can such that the the number of iterations of the while loop in the Gale-Shapley algorithm is $O(f(n))$ on any input of n hospitals and n students.

4. (5 points) Write the name or definition of the following set symbols:

   (a) $\subseteq$:

   (b) $\cup$:

   (c) $\cap$:

   (d) $\mathscr{P}$:

   (e) $\in$:

Section 2 (Algorithm Analysis)

5. (5 points) Evaluate the following:

   (a) $\log_2(16) =$

   (b) $2^{\log_2(123456789)} =$

6. (15 points) Recall the definition of big O: a function $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 1$ such that for every $n \geq n_0$, $f(n) \leq cg(n)$.

   Use this definition to prove that $3n^2 + 5n + 10$ is $O(n^2)$.

7. (10 points) Give a function $f(n)$ such that the worst-case runtime of the following algorithm is $\Theta(f(n))$. Recall that $\lfloor x \rfloor$ takes the *floor* of x, meaning that it rounds down to the nearest integer.

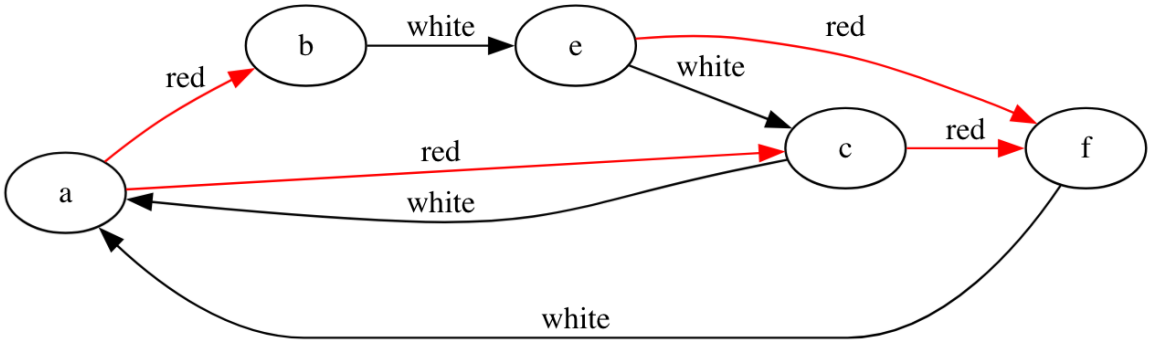
```
Algorithm-1(array A of length n):
    Result = 0
    For i in 1 to n:
        If A[i] is even:
            For j in i + ⌊n/2⌋ to n:
                Add A[j] to Result
    Return Result
```

Section 3 (Graph Algorithms)

Similar to a *French flag walk*, a *Polish flag walk* is a walk in a directed graph where the sequence of edge colors is red, white, red, and so on. More formally, a walk $v_0 \to v_1 \to \ldots \to v_k$ is a Polish flag walk if, for every integer $i \in 0, \ldots, k-1$, the edge $v_i \to v_{i+1}$ is:

- red if $i \bmod 2 = 0$, and
- white if $i \bmod 2 = 1$.

8. (5 points) What is the set of nodes reachable from node a by a Polish flag walk in the following graph?



We now define a new graph G' such that the nodes reachable by a Polish flag walk in G are exactly the nodes reachable by any walk in G'.

We create the vertex set V' of G' as follows: for each vertex $v \in V$, we create two vertices $(v, 0), (v, 1)$ in V'. We then create the edge set E' of G' as follows: for each edge $(u, v) \in E$:

- if (u, v) is red, we add the edge $((u, 0), (v, 1))$ to E', and
- if (u, v) is white, we add the edge $((u, 1), (v, 0))$ to E'.

9. (13 points) Draw G' for the graph above.

10. (6 points) Which of the following algorithms, applied to G', would find all nodes reachable by a Polish flag walk from node a in G? (You may select more than one answer.)

    (a) Depth-first search starting at node $(a, 0)$

    (b) Depth-first search starting at node $(a, 1)$

    (c) Breadth-first search starting at node $(a, 0)$

    (d) Breadth-first search starting at node $(a, 1)$

    (e) Topological sort beginning at node $(a, 0)$

    (f) Topological sort beginning at node $(a, 1)$

11. (6 points) An *Indonesian flag walk* is a walk in a directed graph where the sequence of edge colors is white, red, white, red, etc. Which of the following algorithms would find all nodes reachable by an Indonesian flag walk from node a in G? (You may select more than one answer.)

    (a) Depth-first search starting at node $(a, 0)$

    (b) Depth-first search starting at node $(a, 1)$

    (c) Breadth-first search starting at node $(a, 0)$

    (d) Breadth-first search starting at node $(a, 1)$

    (e) Topological sort beginning at node $(a, 0)$

    (f) Topological sort beginning at node $(a, 1)$

Section 4 (Greedy Graph Algorithms)

Recall Dijkstra's algorithm to compute the shortest path distance from one node s to all other nodes in a directed graph with non-negative edge weights:

> Dijkstra(directed graph $G = (V, E)$ with non-negative edge weights; node s)
> Let S be the set of explored nodes
> For each $u \in S$, we store a distance $d(u)$
> Initially $S = \{s\}$ and $d(s) = 0$
> While $S \neq V$:
> Select a node $v \notin S$ with at least one edge from S for which
> $d'(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e$ is as small as possible
> Add v to S and define $d(v) = d'(v)$

Fill in the following blanks to give a proof by induction for the following claim.

Claim: Given a directed graph G with non-negative edge weights and a starting node s, Dijkstra's algorithm labels every node $v \in V$ with $d(v)$ equal to the shortest path distance from s to v.

Proof:

Let $G = (V, E)$ be a directed graph with non-negative edge weights and let $s \in V$ be a starting node.

(3 points) Inductive hypothesis (you fill in; do induction over number of nodes in G):

There are two cases:

(1 point) Base case (use $|V| = 1$):

(6 points) Inductive case: Assume $V > 1$. Let v be the last node processed by Dijkstra's algorithm. Let $G' = (V', E')$ be the graph obtained by removing v and all edges incident to v from G (you fill in the rest)