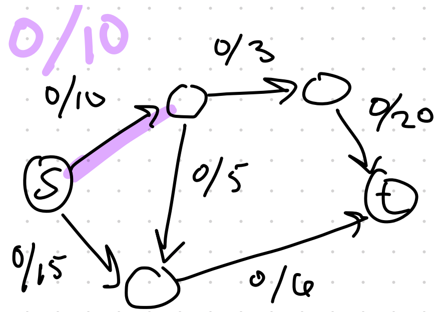
Given

$G = (V, E)$

$s \in V$, $t \in V$

$c : E \to \mathbb{R}^{\geq 0}$



Max flow problem:

find flow $f : E \to \mathbb{R}^{\geq 0}$ satisfying

- $\forall v \in V \setminus \{s, t\} : \sum_u f(u \to v) = \sum_w f(v \to w)$
  conservation
- $\forall e \in E : 0 \leq f(e) \leq c(e)$
  feasibility
- $|f| = \sum_w f(s \to w)$ is maximized

Min cut problem:

find partition S, T so that

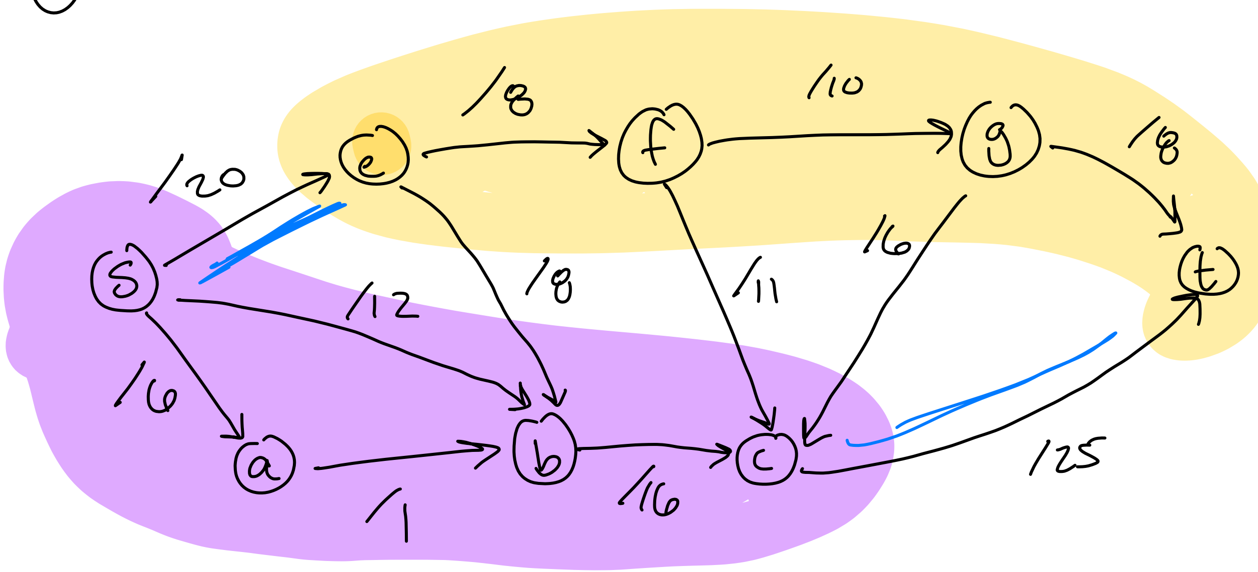
$\|S, T\| = \sum_{v \in S} \sum_{w \in T} c(v \to w)$ is minimized.

capacity of cut S, T

let $S = \{s, a, b, c\}$

(1) what is T? $T = \{e, f, g, t\}$

(2) what is $\|S, T\|$? $= 45$



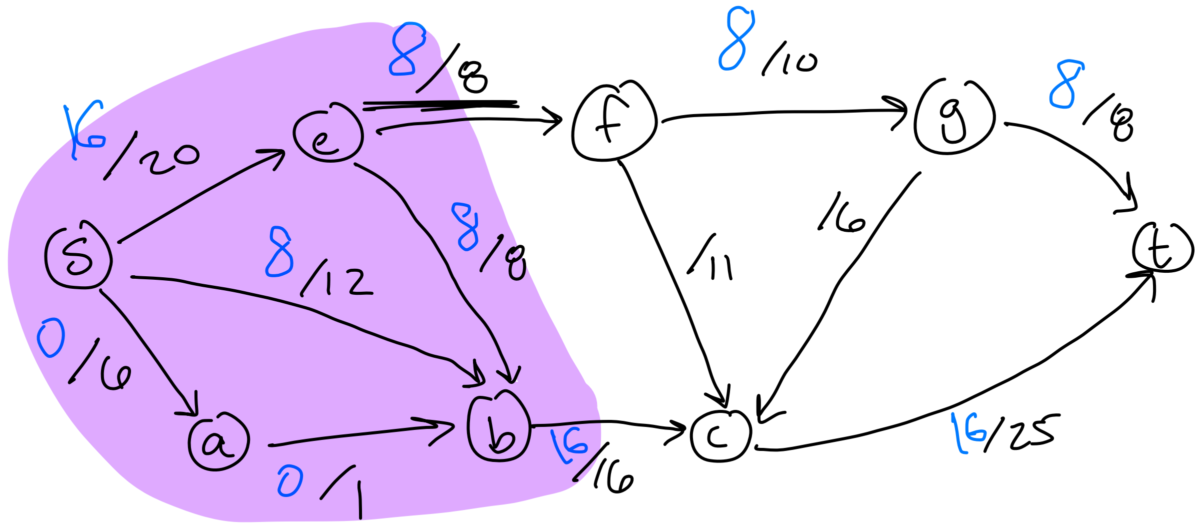
Let $S = \{s, a, b, c\}$

① what is T?

② what is $\|S, T\|$?

$|f| = 24$

$\|S, T\| = 24$

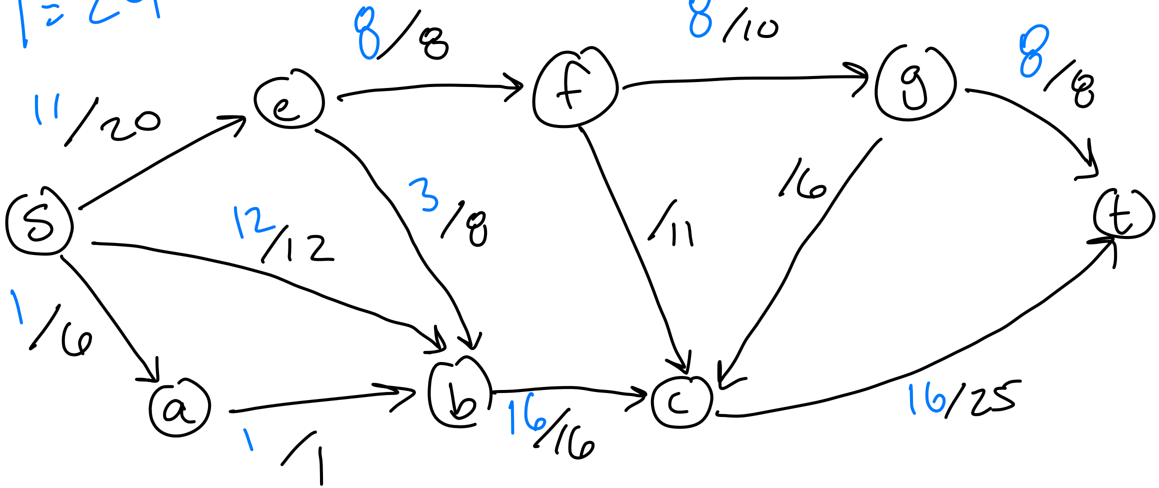


Ⓕ what is the maximum flow – that is, $f : E \to \mathbb{R}^{\geq 0}$ s.t. $|f|$ is maximized?

Ⓒ what is the minimum cut – that is, a new partition S, T so that $\|S, T\|$ is minimized?

GreedyFlow:
while ∃ path from s to t:
push max amnt of flow on that path

$|f| = 24$



Theorem $\max |f| = \min \|S,T\|$.

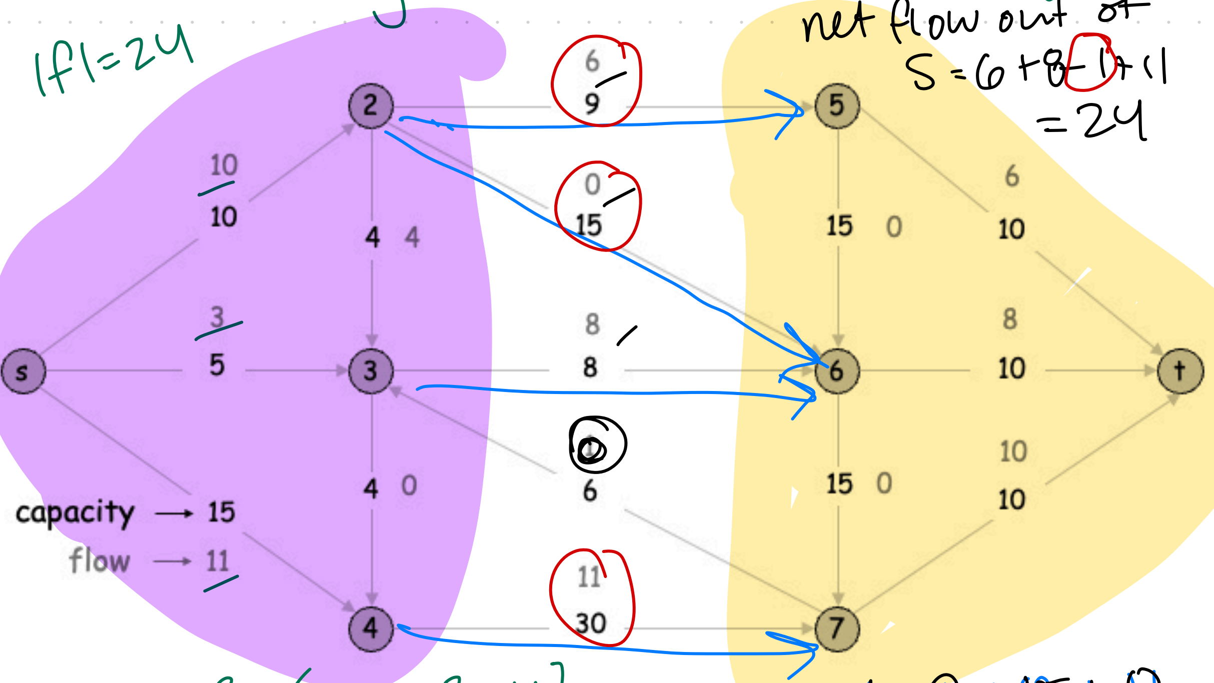
Proof sketch

Part 1: we show $|f| \le \|S,T\|$ for any f, S, T.

Let f be any flow and S,T be any cut.

$|f| = 24$

net flow out of $S = 6 + 8 - 1 + 11$
$= 24$



$S = \{s, 2, 3, 4\}$

$\|S,T\| = 9 + 15 + 8 + 30$

$|f| =$ net flow out of S

$\le$ flow out of S

$\le$ capacity out of S

$= \|S,T\|$

conservation of flow and internal edges

removing negatives

feasibility of flow

def. of $\|S,T\|$

$|f| \le \|S,T\|$

suppose $|f| = \|S,T\|$.
Then $|f|$ must be maximum and $\|S,T\|$ must be minimum.

what about a flow f would make $|f| = \|S,T\|$?

- no flow from T to S
- edges from S to T are saturated

Part 2: Show that there always is f, S, T, such that $|f| = \|S,T\|$.

let f be a flow.

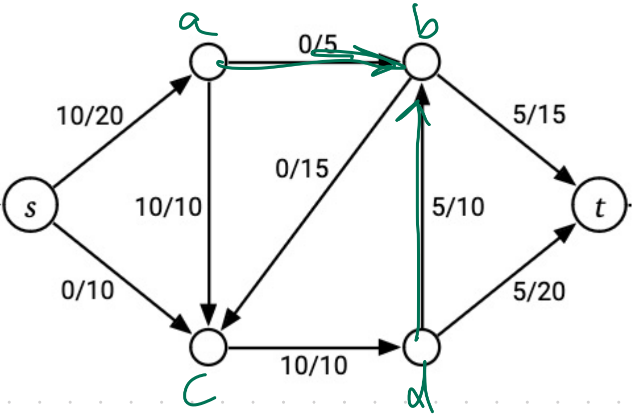
let $c_f = V \times V \to \mathbb{R}^{\geq 0}$

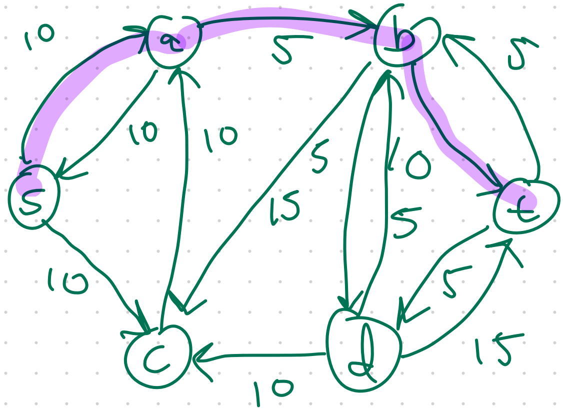
"residual capacity"

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(u \to v) & \text{if } v \to u \in E \end{cases}$$

$$c_f(b \to t) = c(b \to t) - f(b \to t) = 15 - 5 = 10$$

$$c_f(t \to b) = f(b \to t) = 5$$

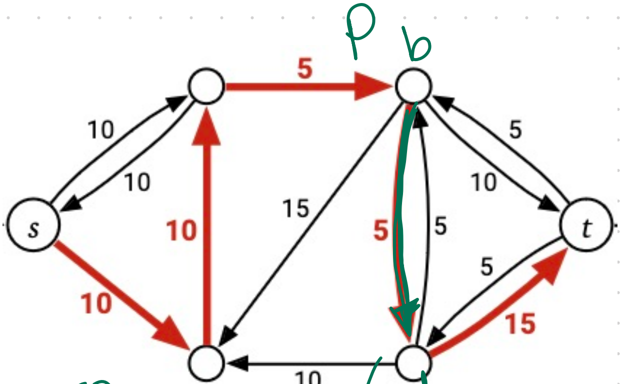




Call any path from s to t an augmenting path.

G_f

Case I: there is an augmenting path.

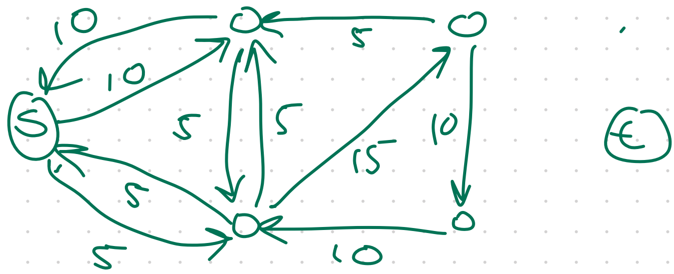


push the max. amnt of flow through it.

Claim: adding P to f w/ max flow is a valid flow.

1. forward ✓
2. reverse ✓

|new flow| > |f|


10
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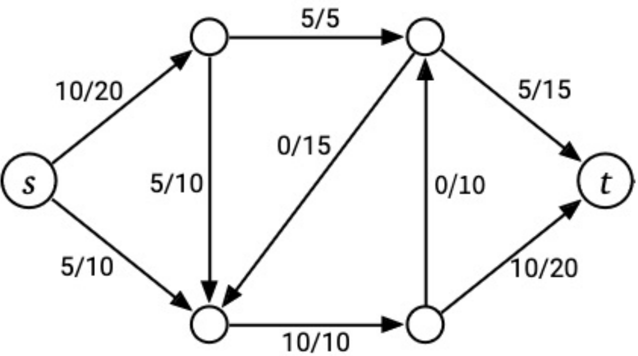
5/5
10/20
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s
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Technique	Direct	With dynamic trees	Source(s)
Blocking flow	$O(V^2E)$	$O(VE\log V)$	[Dinitz; Karzanov; Even and Itai; Sleator and Tarjan]
Network simplex	$O(V^2E)$	$O(VE\log V)$	[Dantzig; Goldfarb and Hao; Goldberg, Grigoriadis, and Tarjan]
Push-relabel (generic)	$O(V^2E)$	—	[Goldberg and Tarjan]
Push-relabel (FIFO)	$O(V^3)$	$O(VE\log(V^2/E))$	[Goldberg and Tarjan]
Push-relabel (highest label)	$O(V^2\sqrt{E})$	—	[Cheriyan and Maheshwari; Tunçel]
Push-relabel-add games	—	$O(VE\log_{E/(V\log V)} V)$	[Cheriyan and Hagerup; King, Rao, and Tarjan]
Pseudoflow	$O(V^2E)$	$O(VE\log V)$	[Hochbaum]
Pseudoflow (highest label)	$O(V^3)$	$O(VE\log(V^2/E))$	[Hochbaum and Orlin]
Incremental BFS	$O(V^2E)$	$O(VE\log(V^2/E))$	[Goldberg, Held, Kaplan, Tarjan, and Werneck]
Compact networks	—	$O(VE)$	[Orlin]