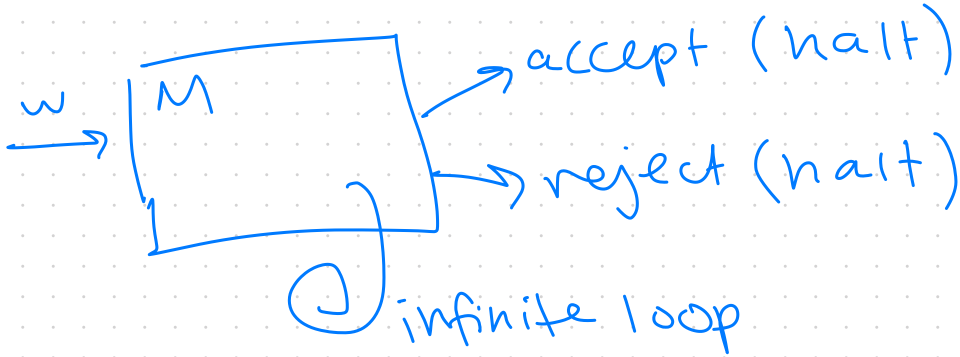
undecidable = no algorithm at all

Given input w TM M:



M defines 4 languages:

- $\text{ACCEPT}(M) = \{w \in \Sigma^* : M \text{ accepts}\}$
- $\text{REJECT}(M) = \{w \in \Sigma^* : M \text{ rejects}\}$
- $\text{HALT}(M) = \{w \in \Sigma^* : M \text{ halts}\}$
  $= \text{ACCEPT}(M) \cup \text{REJECT}(M)$
- $\text{DIVERGE}(M) = \Sigma^* \setminus \text{HALT}(M)$

Let $\langle\ \rangle$ be some encoding scheme for TMs.

$\langle M \rangle$ = encoding of M

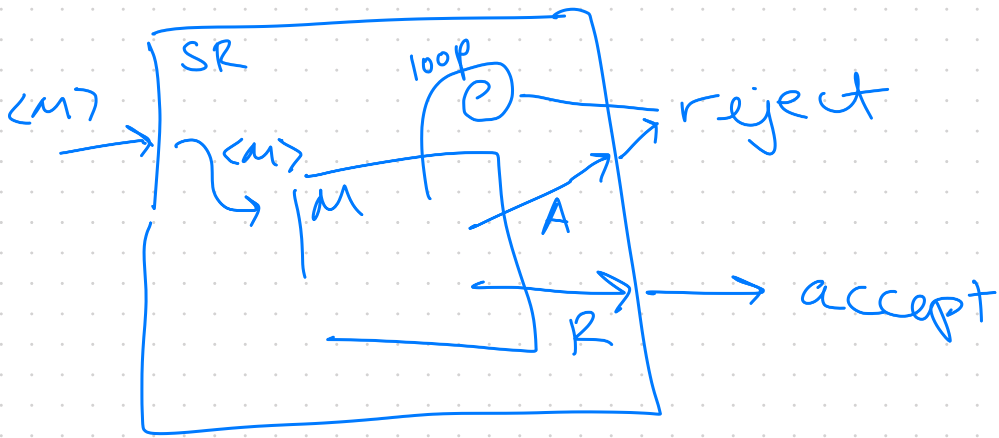
A language is <u>decidable</u> if there is a decider for it.

Let SELFREJECT = $\{\langle M\rangle : M \text{ rejects } \langle M\rangle\}$

Theorem: SELFREJECT is undecidable.

Proof: Suppose not

Let SR be the TM that decides SELFREJECT.



ACCEPT(SR) = SELFREJECT
DIVERGE(SR) = $\emptyset$

SR accepts $\langle M\rangle$ iff M rejects $\langle M\rangle$

SR accepts $\langle SR\rangle$ iff SR rejects $\langle SR\rangle$

$p \iff \neg p$

Contradiction. So SR can't exist.

TM Y. $Q = \{q_1, q_{acc}, q_{reject}, q_{start}\}$

$\Gamma = \{0, 1, \square, x, \$\}$

δ ...

$\# q_1 \# q_{acc} \# \ldots \# \$ \delta(q_1, 0) = (q_1, 1, +1) \$$

$\ldots \$$

HALT = $\{\langle M, w\rangle : M$ halts on input $w\}$

can we just run M on w?

no - what if M loops on w?

let SELFHALT = $\{\langle M\rangle : M$ halts on $\langle M\rangle\}$

Theorem: SELFHALT is undecidable.

Proof: Suppose not.

Let SH be a TM that decides SELFHALT.

can we be done?

So SH accepts $\langle M\rangle \iff M$ halts on $\langle M\rangle$

So SH accepts $\langle SH\rangle \iff SH$ halts on $\langle SH\rangle$

nope

Let SH^X be a TM built from SH where every transition to an accept state is redirected to a hang state.

SH^X does not halt on $\langle M\rangle \iff M$ halts on $\langle M\rangle$

SH^X does not halt on $\langle SH^X\rangle \iff SH^X$ halts on $\langle SH^X\rangle$

Contradiction! So SH can't exist.

Theorem: HALT is undecidable.

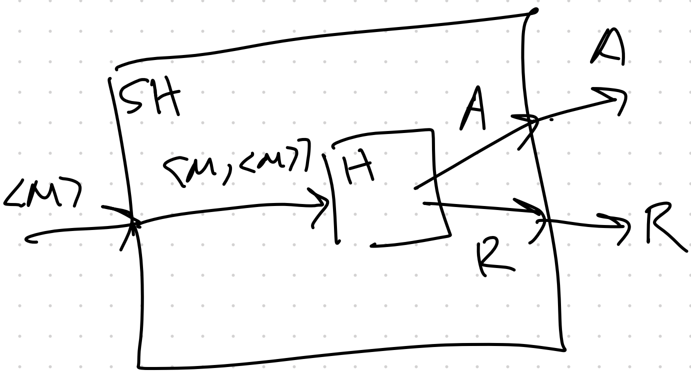
Proof: Suppose HALT is decidable.

Let H be a TM that decides HALT.

That is, H accepts $\langle M, w\rangle \iff M$ halts on w.

[Now, we can decide SELFHALT by running $\langle M, \langle M\rangle\rangle$ on H.]

But SELFHALT is undecidable, so H can't exist.



Recall SELFHALT = $\{\langle M\rangle : M \text{ halts on } \langle M\rangle\}$

Proof by reduction that X is undecidable

If you can reduce an undecidable problem to X, then X is undecidable.