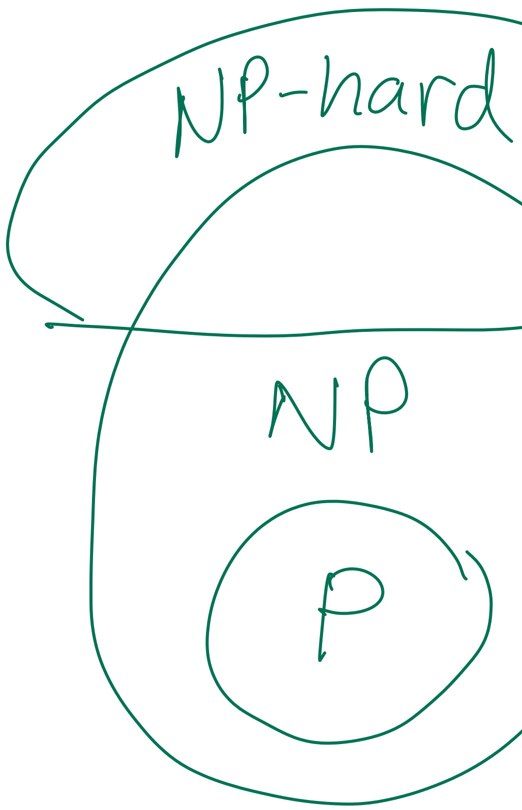
Handling NP-Hardness



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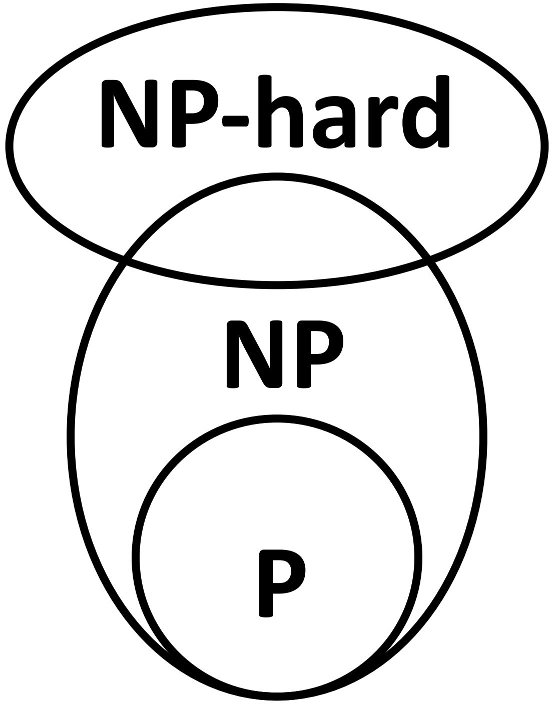

Your problem is NP-hard. What to do?

1. Brute Force — exponential runtime
2. Heuristics — no guarantee on optimality
3. Approximation Algorithms

Approximation Algorithms

$$ALG \leq \alpha \, OPT$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

Approximation Algorithms

1.25

$$ALG \leq \alpha \; OPT$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

Example: If my CheapestToiletPaperInMissoula algorithm is a 1.25-approximation algorithm, the cost of the toilet paper it finds is at most 1.25 times the optimal cost.

Approximation Algorithms

2.50   1.25 · 2.00

$$ALG \leq \alpha \, OPT$$

Cost (size) of algorithm's solution.

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Example: If my CheapestToiletPaperInMissoula algorithm is a 1.25-approximation algorithm, the cost of the toilet paper it finds is at most 1.25 times the optimal cost.

I.e. If cheapest toilet paper in Missoula is $2.00/roll, CheapestToiletPaperInMissoula will find toilet paper that is at most $ /roll.

Approximation Algorithms

$$ALG \leq \alpha \, OPT$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

Example: If my CheapestToiletPaperInMissoula algorithm is a 1.25-approximation algorithm, the cost of the toilet paper it finds is at most 1.25 times the optimal cost.

I.e. If cheapest toilet paper in Missoula is \$2.00/roll, CheapestToiletPaperInMissoula will find toilet paper that is at most \$2.50/roll.

Approximation Algorithms

$$\text{ALG} \leq \alpha\ \text{OPT}$$

Cost (size) of algorithm's solution. → ALG

Approximation Ratio → α

Cost (size) of optimal solution. → OPT

Example:

Approximation Algorithms

$$\text{ALG} \leq \alpha\ \text{OPT}$$

- ALG: Cost (size) of algorithm's solution.
- α: Approximation Ratio
- OPT: Cost (size) of optimal solution.

Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.

Approximation Algorithms

$$\text{ALG} \leq \alpha \ \text{OPT}$$

Cost (size) of algorithm's solution. → ALG

Approximation Ratio → α

Cost (size) of optimal solution. → OPT

Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost (size) 746.125.

Approximation Algorithms

$$\text{ALG} \leq \alpha \text{ OPT}$$

Cost (size) of algorithm's solution. — Approximation Ratio — Cost (size) of optimal solution.

Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost (size) 746.125.

What do I know about OPT?

Approximation Algorithms

$$\text{ALG} \leq \alpha \text{ OPT}$$

Cost (size) of algorithm's solution.

Approximation Ratio

Cost (size) of optimal solution.

Example:

- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost (size) 746.125.

Then, I know that $746.125 \leq 1.12$ OPT

Approximation Algorithms

$$ALG \leq \alpha\, OPT$$

Cost (size) of algorithm's solution. — Approximation Ratio — Cost (size) of optimal solution.

Example:

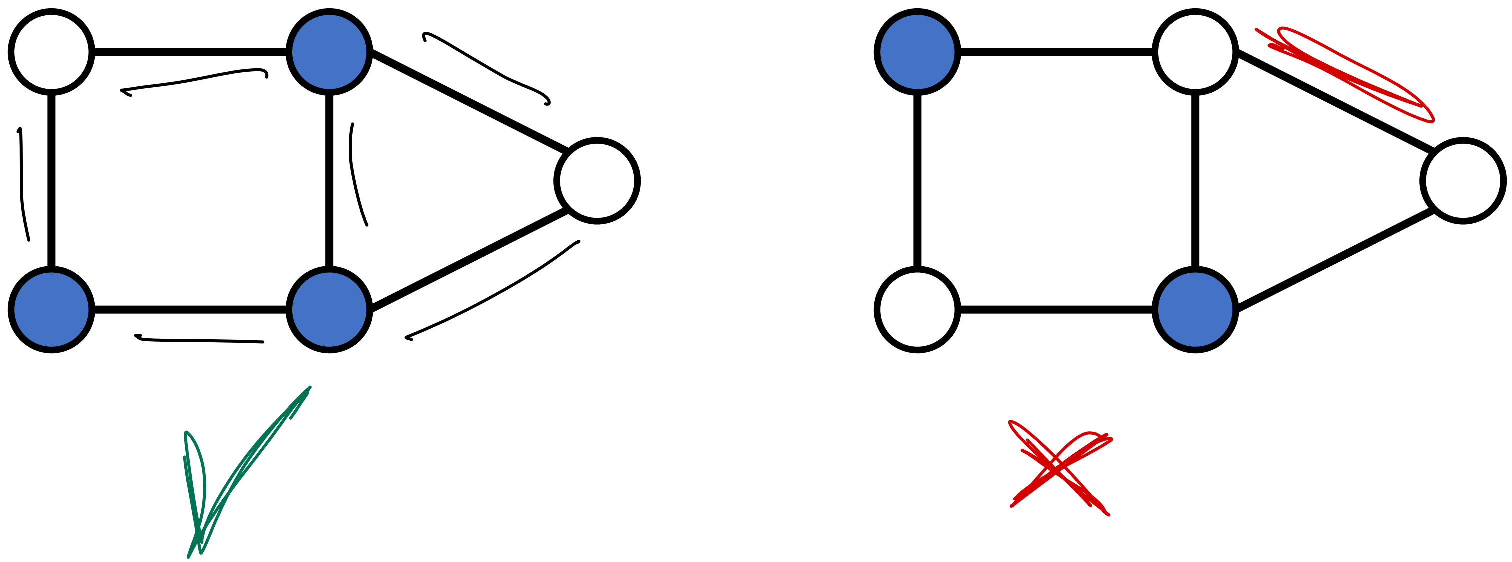
- Suppose I know my algorithm is a 1.12-approximation algorithm.
- Suppose my algorithm returns a solution of cost (size) 746.125.

Then, I know that $746.125 \leq 1.12\ OPT$

$$\Rightarrow \frac{746.125}{1.12} = 666.183 \leq OPT \leq 746.125$$

Vertex Cover – Problem

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.



Vertex Cover

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```
VC algorithm:

        ???
```

Vertex Cover

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VC 2-approximation algorithm:
  while uncovered edge exists
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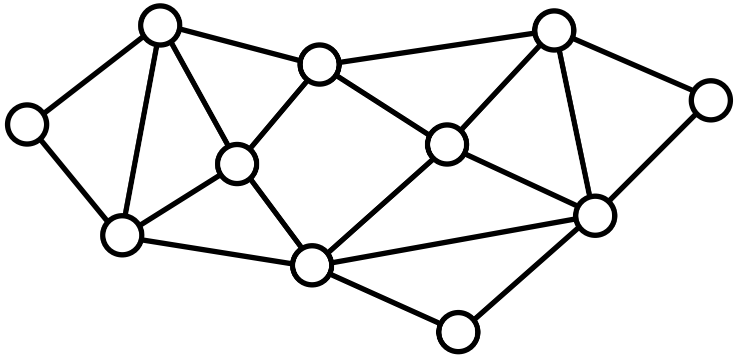
An edge is uncovered if it does not share vertices with any previously selected edges.



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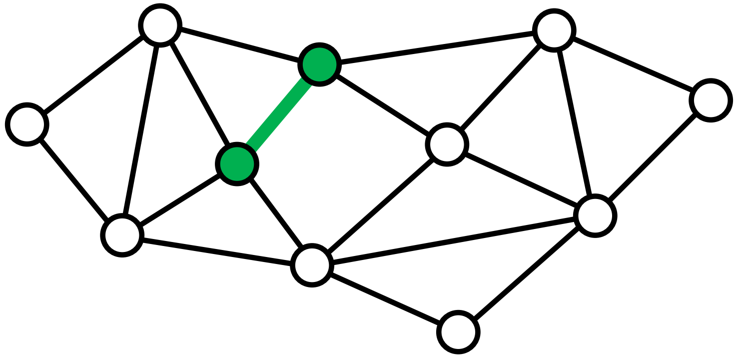

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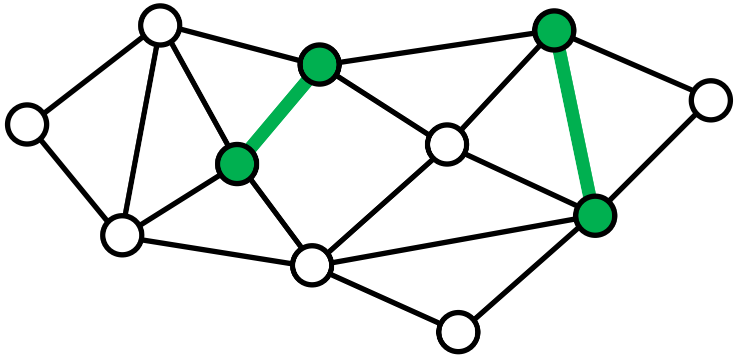

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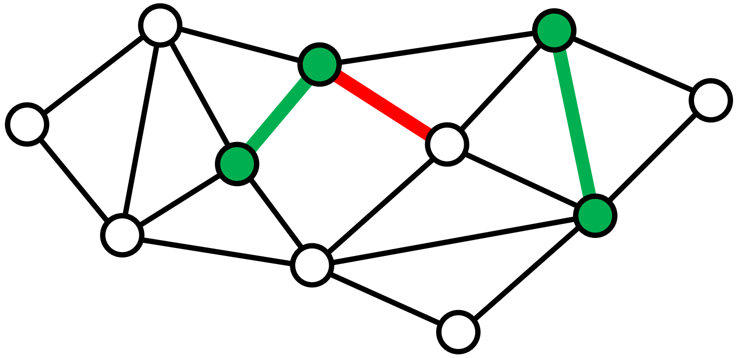

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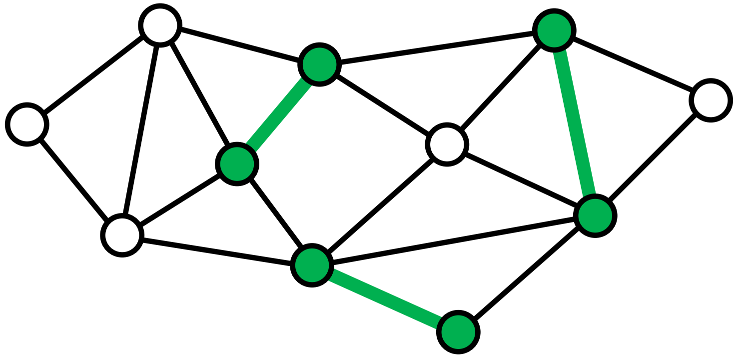

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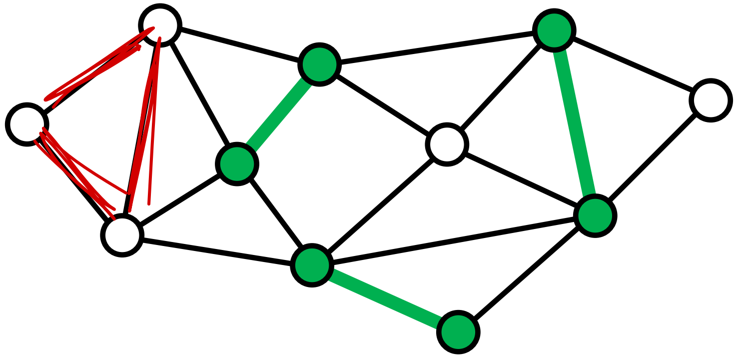
Which edge gets selected next?



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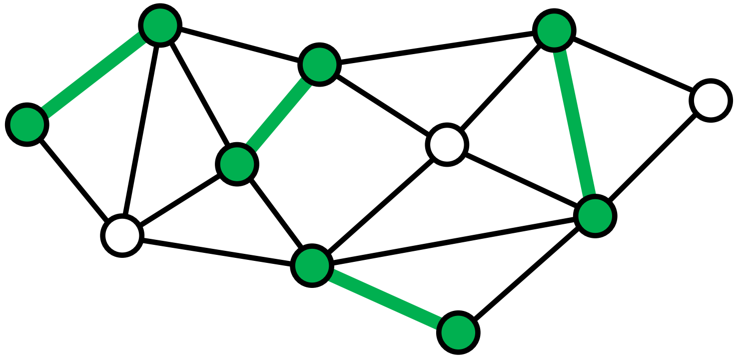

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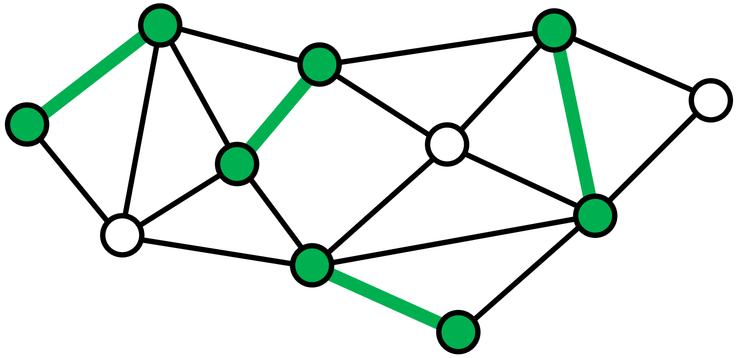

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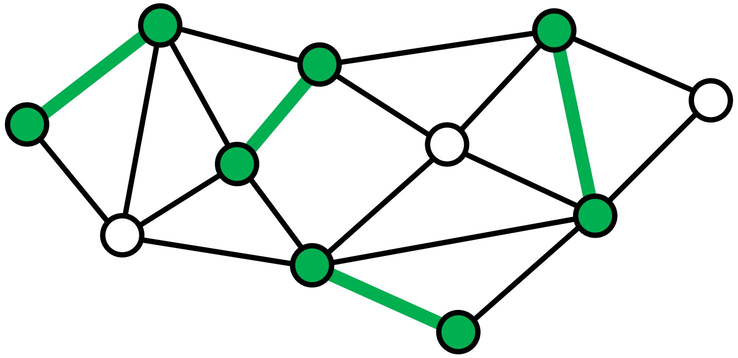
$\Rightarrow$ # vertices selected by algorithm = ALG = ??



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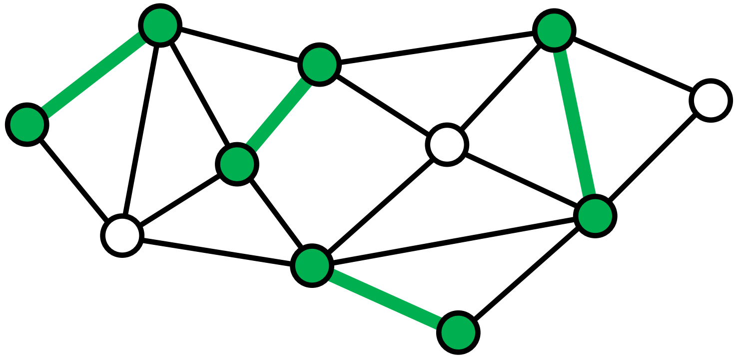
ALG ≤ α OPT

An edge is uncovered if it does not share vertices with any previously selected edges. Let ***E'*** be the edges selected by the algorithm.

⇒ # vertices selected by algorithm = ALG = ??

Discuss with a partner



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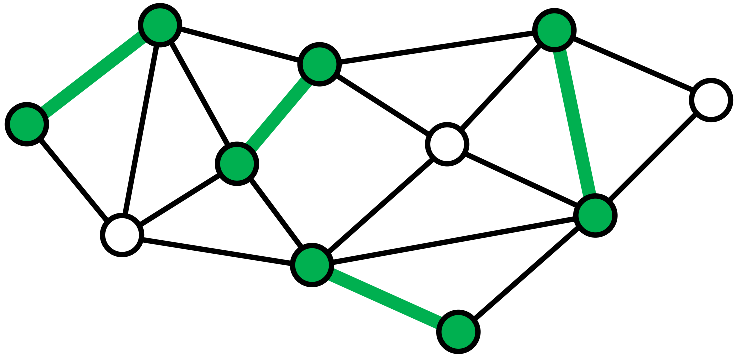
An edge is uncovered if it does not share vertices with any previously selected edges. Let ***E'*** be the edges selected by the algorithm.

$\Rightarrow$ # vertices selected by algorithm = ALG = 2 |***E'***|



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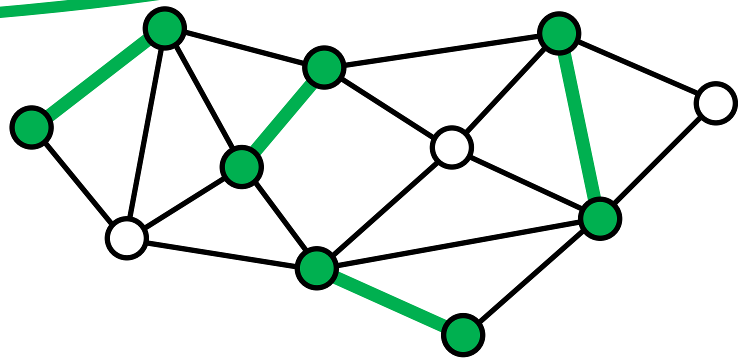
A vertex from each edge in ***E'*** must be part of *every* vertex cover.

True or False?

Discuss with a partner



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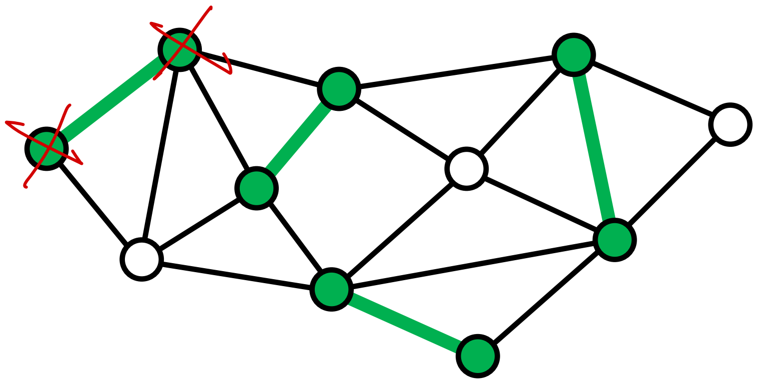
$\Rightarrow$ # vertices selected by algorithm = ALG = 2 |***E'***|

A vertex from each edge in ***E'*** must be part of *every* vertex cover.

If we selected fewer than one vertex per edge, we would not have a vertex cover, because that edge would not be covered!



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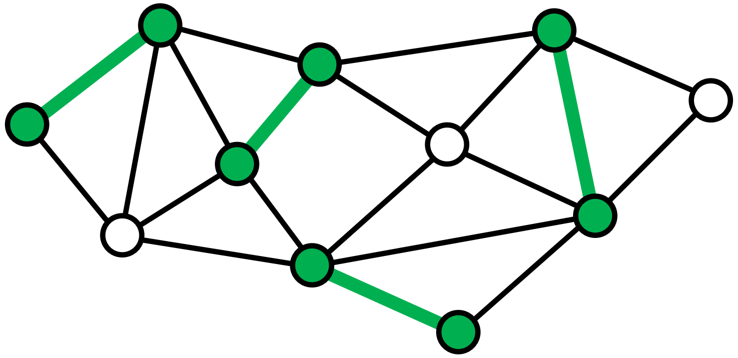
$\Rightarrow$ In relation to OPT??



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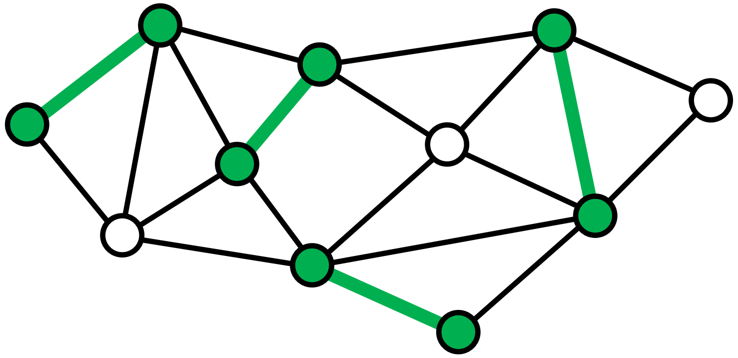
$\Rightarrow$ # vertices selected by algorithm = ALG = 2 $|E'|$

A vertex from each edge in E' must be part of <u>*every*</u> vertex cover.

$\Rightarrow |E'| \leq$ OPT



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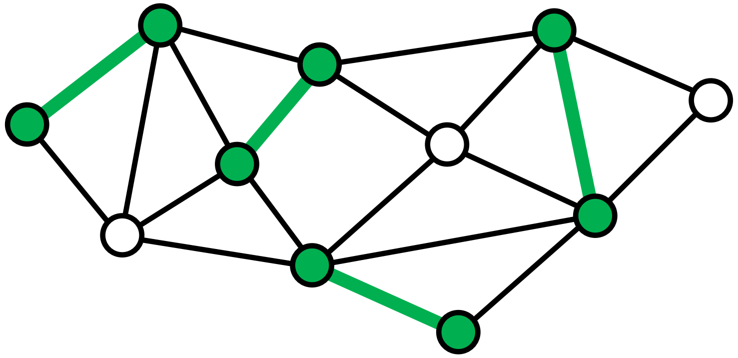
$\Rightarrow$ # vertices selected by algorithm = ALG = 2 |***E'***|

A vertex from each edge in ***E'*** must be part of *every* vertex cover.

$\Rightarrow$ |***E'***| $\leq$ OPT

…ALG ≤ α OPT??



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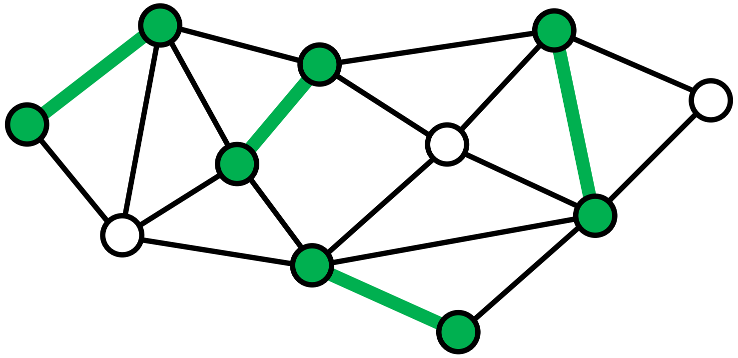
$\Rightarrow$ # vertices selected by algorithm = ALG = 2 $|E'|$

A vertex from each edge in E' must be part of *every* vertex cover.

$\Rightarrow |E'| \leq$ OPT

Therefore, ALG = 2 $|E'|$



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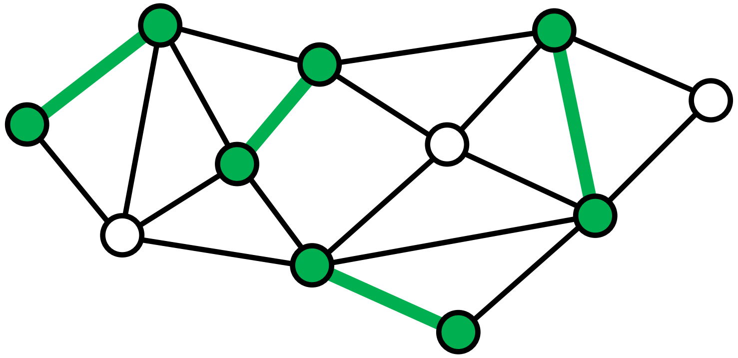
A vertex from each edge in ***E'*** must be part of *every* vertex cover.

$\Rightarrow |E'| \leq$ OPT

Therefore, ALG = 2 |***E'***| $\leq$ 2 OPT



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An edge is uncovered if it does not share vertices with any previously selected edges. Let ***E'*** be the edges selected by the algorithm.

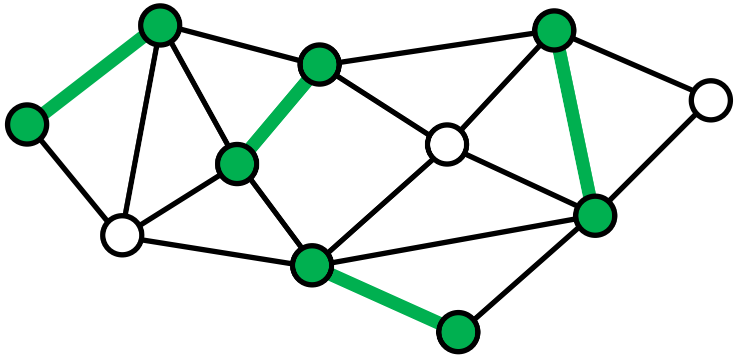
$\Rightarrow$ # vertices selected by algorithm = ALG = 2 |***E'***|

A vertex from each edge in ***E'*** must be part of *every* vertex cover.

$\Rightarrow$ |***E'***| $\leq$ OPT

Therefore, ALG = 2 |***E'***| $\leq$ 2 OPT $\Longrightarrow$ ALG $\leq$ 2 OPT

$\alpha = 2$



Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

$\Longrightarrow \text{ALG} \leq 2\ \text{OPT}$

Is this the best this algorithm can do? $\text{ALG} \leq 1.5\ \text{OPT}$? ✗

$\text{ALG} = 2\ \text{OPT}$

Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

$$\Longrightarrow \text{ALG} \leq 2\ \text{OPT}$$

Is this the best this algorithm can do?

I.e. Can we guarantee this algorithm does better than 2 OPT?

Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

$\Longrightarrow \text{ALG} \leq 2\ \text{OPT}$

Is this the best this algorithm can do?

I.e. Can we guarantee this algorithm does better than 2 OPT?

Is there a graph where this algorithm does exactly 2 OPT?

Which of these would be easier to prove?

Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

$$\implies \text{ALG} \leq 2\ \text{OPT}$$

Is this the best this algorithm can do?

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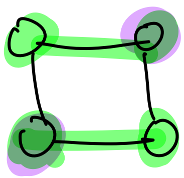
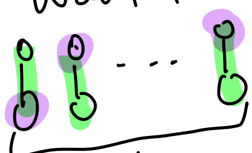
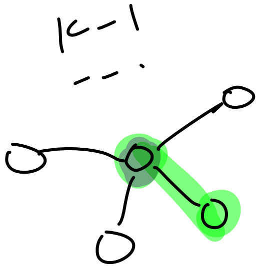
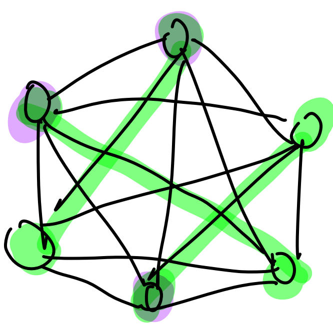
Is there a graph where this algorithm does exactly 2 OPT?

Try to find a graph where ALG = 2 OPT for this algorithm

G	ALG	OPT	α
	2	1	2
	4	2	2
Want k verts  k/2 disjoint edges	k	k/2	2

want k verts, connected G	ALG	OPT	α
	2	1	2
	6	3	2

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$$\Longrightarrow \text{ALG} \leq 2\ \text{OPT}$$

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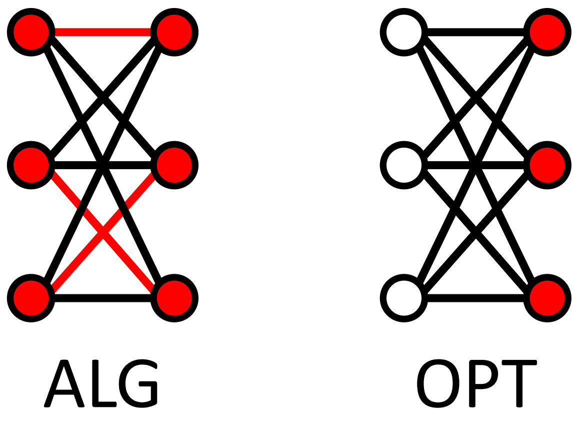
Is there a graph where this algorithm does exactly 2 OPT?

Complete Bipartite Graph



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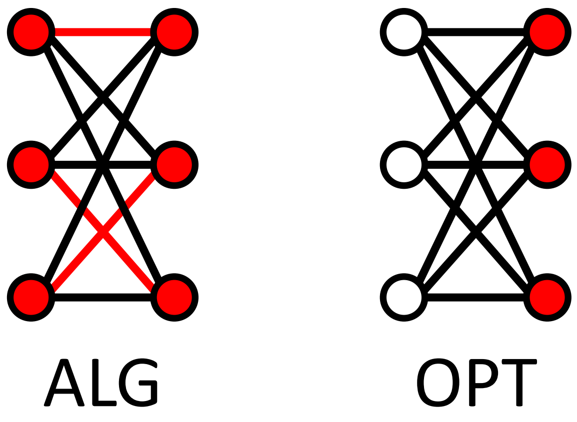

$\underline{|\text{ALG}| = 2k}$: $\underline{v \notin \text{ALG} \Rightarrow}$ all neighbors are $\Longrightarrow$ k edges selected $\Longrightarrow$ all $2k$ nodes selected.

$\underline{|\text{OPT}| = k}$: Fewer than $\underline{k}$ nodes selected $\Longrightarrow$ $\exists$ unselected edge.

Vertex Cover – Improvement

```
while uncovered edge exists
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```

$\Longrightarrow$ ALG $\leq$ 2 OPT

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I.e. Can we guarantee this algorithm does better than 2 OPT?

Is there a graph where this algorithm does exactly 2 OPT?

T or F ?

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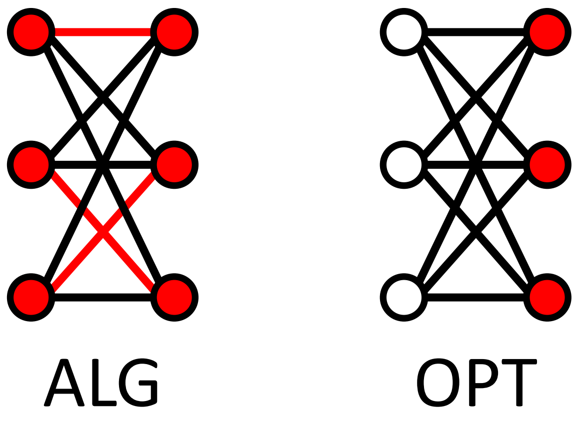

∴ The best Vertex Cover can be approximated is within a factor of 2

Vertex Cover – Improvement

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while uncovered edge exists
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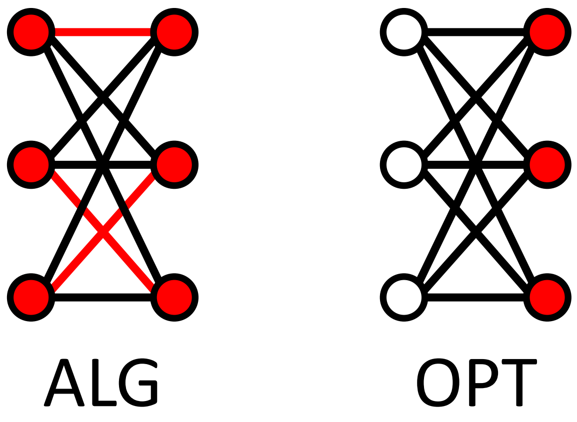
Complete Bipartite Graph



The best Vertex Cover can be approximated is within a factor of 2

True or false?

Vertex Cover – Improvement

```
while uncovered edge exists
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$\implies$ ALG $\leq$ 2 OPT

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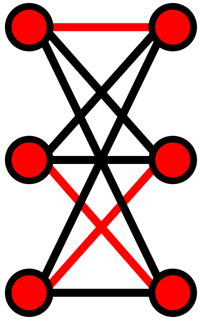
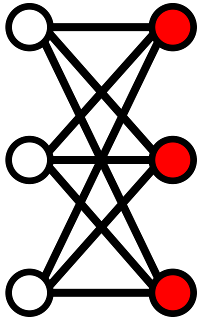
Is there a graph where this algorithm does exactly 2 OPT?

Complete Bipartite Graph



ALG



OPT

~~∴ The best Vertex Cover can be approximated is within a factor of 2~~

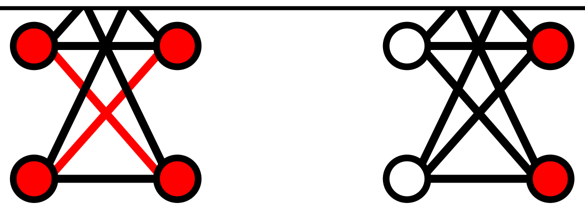
Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

VC Inapproximability:

- Cannot be approximated within a factor of 1.3606 unless P=NP.

Complete Bipartite Graph



ALG OPT

~~can be approximated is within a factor of 2~~

Vertex Cover – Improvement

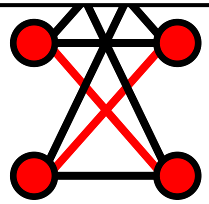
```
while uncovered edge exists
   select both vertices from uncovered edge
```

VC Inapproximability:

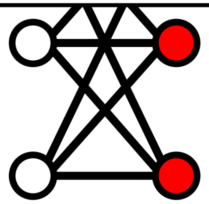
- Cannot be approximated within a factor of 1.3606 unless P=NP.

How do you think we prove this?

Complete Bipartite Graph



ALG



OPT

can be approximated is within a factor of 2

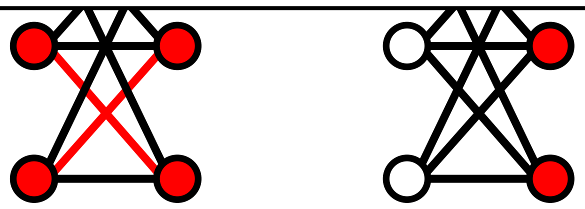
Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

VC Inapproximability:

- Cannot be approximated within a factor of 1.3606 unless P=NP.
- Cannot be approximated within any constant factor better than 2 unless the Unique Games Conjecture is false.

Complete Bipartite Graph



ALG OPT

can be approximated is within a factor of 2

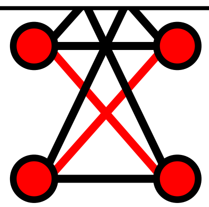
Vertex Cover – Improvement

```
while uncovered edge exists
   select both vertices from uncovered edge
```

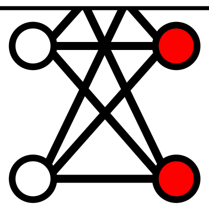
VC Inapproximability:

- Cannot be approximated within a factor of 1.3606 unless P=NP.
- Cannot be approximated within any constant factor better than 2 unless the Unique Games Conjecture is false.
- Is approximable within $2 - \frac{\log \log|V|}{2 \log|V|}$.

Complete Bipartite Graph



ALG



OPT

can be approximated is within a factor of 2