Solving Linear Programs

$x \in \mathbb{R}^d$

input: LP

$$\max \; c^T x \quad \text{s.t.} \quad Ax \le b, \; x \ge 0$$

n constraints

x d-dimensional

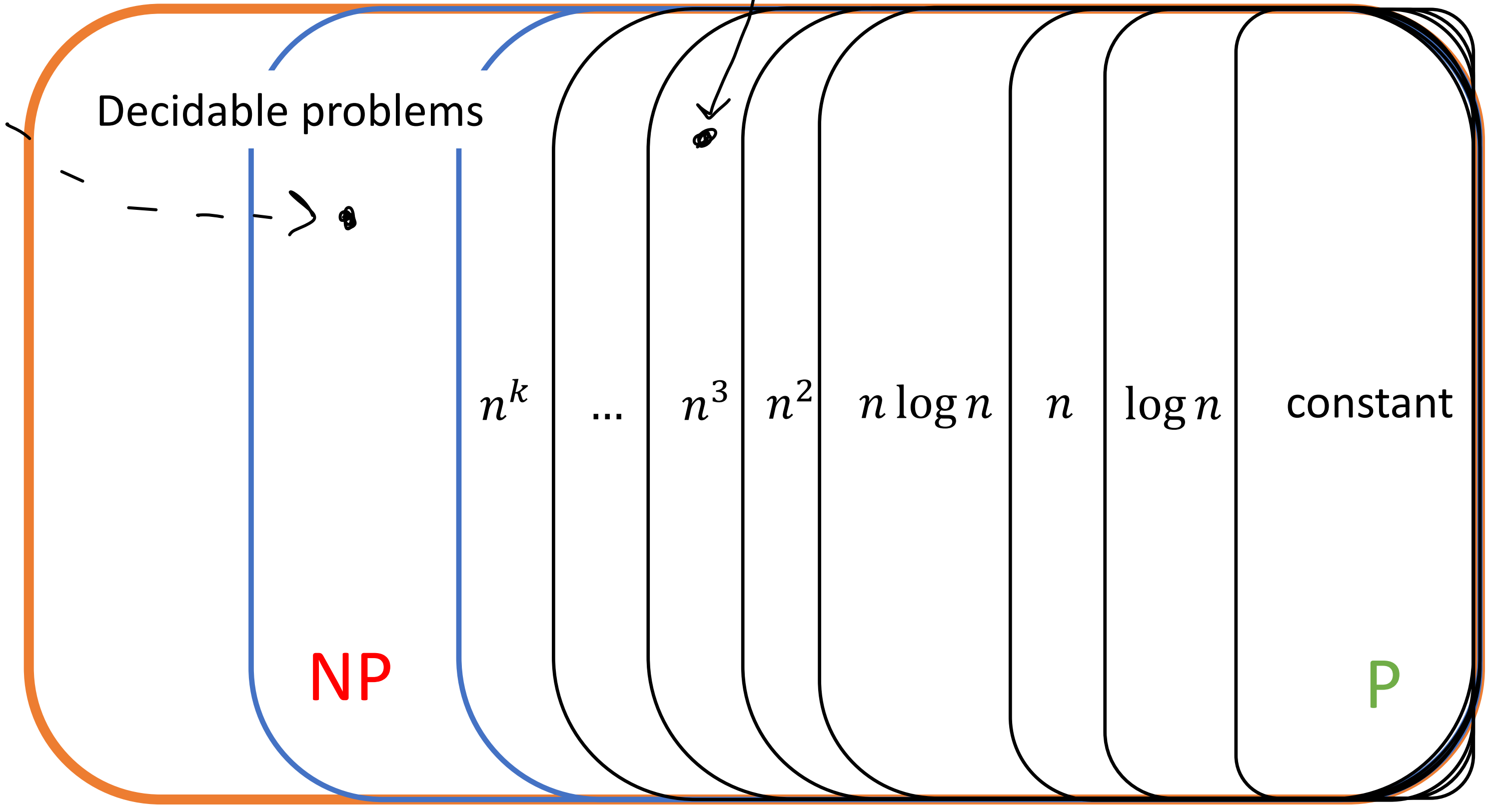
→ n = # constraints + # dims

= 2 · something

output: x^* feasible optimal

best known alg

simplex alg.



Assuming $P \neq NP$

today:

$x \in \mathbb{Z}^d$

↑ integers

Vertex Cover Linear Program

$x \in \mathbb{Z}^d$

Input

Graph $G = (V, E)$
Where $V = \{v_1, v_2, \ldots, v_n\}$
and $\mathrm{E} = \{\{v_i, v_j\}$ where $v_i, v_j \in V\}$

Output

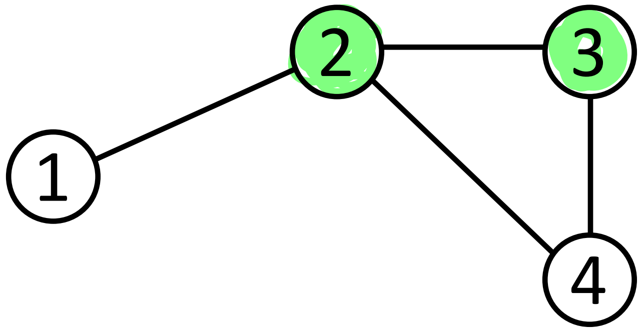
$V' \subseteq V$
so that all edges are covered
for all $\{v_i, v_j\} \in E$, $v_i \in V'$ or $v_j \in V'$
V' as small as possible

Linear Program

Variables: x_i for each v_i
$x_i = 1$ $v_i \in V'$ $x_i \in \{0, 1\}$
$x_i = 0$ $v_i \notin V'$

objective: $\min \sum x_i$

for each edge $\{v_i, v_j\}$:
$x_i + x_j \geq 1$



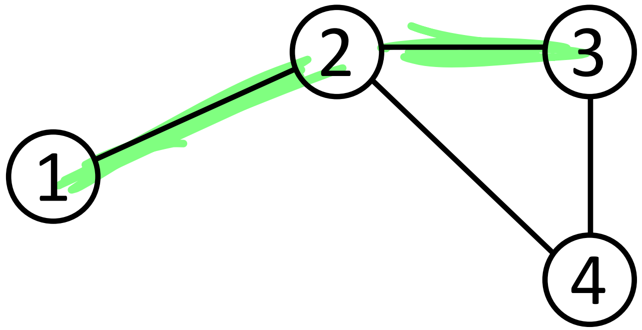
Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $\{v_i, v_j\}$
	$x_i \in \{0,1\}$, for each vertex i

Example:

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 \geq 1$
	$x_2 + x_3 \geq 1$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



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<u>Example:</u>

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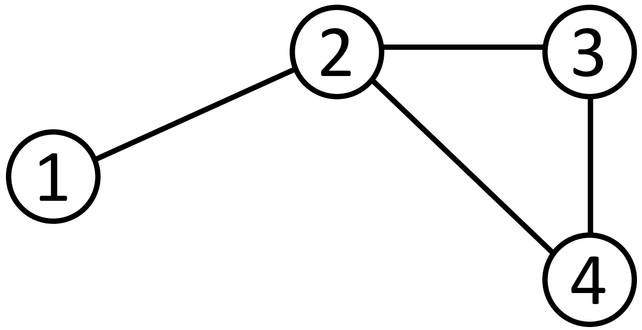

Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

vars: $x_i = 1$ if we select s_i
$x_i = 0$ if not

min $\sum x_i$

$x_1 + x_2 \geq 1$

Vertex Cover example

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $\{v_i, v_j\}$
	$x_i \in \{0,1\}$, for each vertex i

Example:

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 \geq 1$
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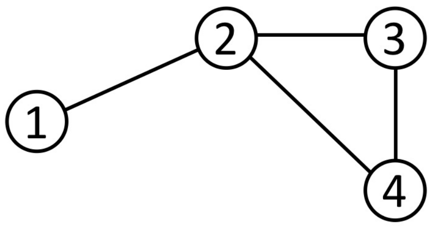

$U = \{1, 4, 7, 8, 10\}$

$S = \left\{ \begin{array}{l} \{1,7,8\}, \{1,4,7\}, \\ \{7,8\}, \{4,8,10\} \end{array} \right\}$

(s_1, s_2, s_3, s_4)

Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

Objective:	$\min \sum_s x_s$
Subject to:	$\sum_{s:\, u \in s} x_s \geq 1$, for each $u \in U$
	$x_s \in \{0,1\}$, for each set s

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \left\{ \begin{matrix} \{1, 7, 8\}, \{1, 4, 7\}, \\ \{7, 8\}, \{4, 8, 10\} \end{matrix} \right\}$$

Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

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Example:

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 \geq 1$
	$x_2 + x_4 \geq 1$
	$x_1 + x_2 + x_3 \geq 1$
	$x_1 + x_3 + x_4 \geq 1$
	$x_4 \geq 1$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

$U = \{1, 4, 7, 8, 10\}$

$S = \left\{ \begin{array}{c} \{1, 7, 8\}, \{1, 4, 7\}, \\ \{7, 8\}, \{4, 8, 10\} \end{array} \right\}$

We now have a reduction from Vertex Cover (Set Cover, ∈NP-Hard) to Solving ILP

We now have a reduction from Vertex Cover to Solving ILP

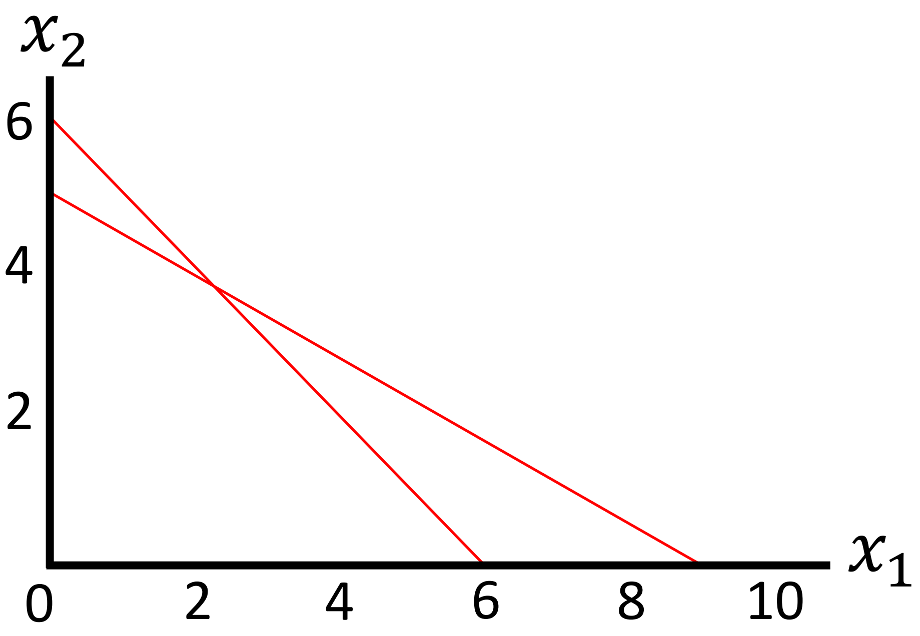
We now have a reduction from Vertex Cover to Solving ILP

So Solving ILP is NP-hard

$x_1, x_2 \in \mathbb{R}$

Objective: $\max 5x_1 + 8x_2$

Subject to:
$$x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, x_2 \geq 0$$



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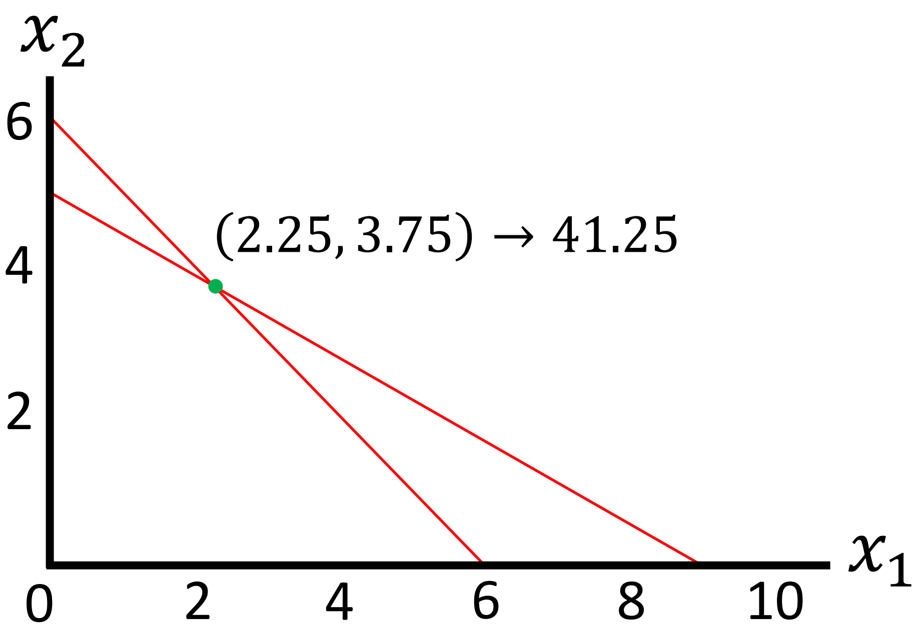

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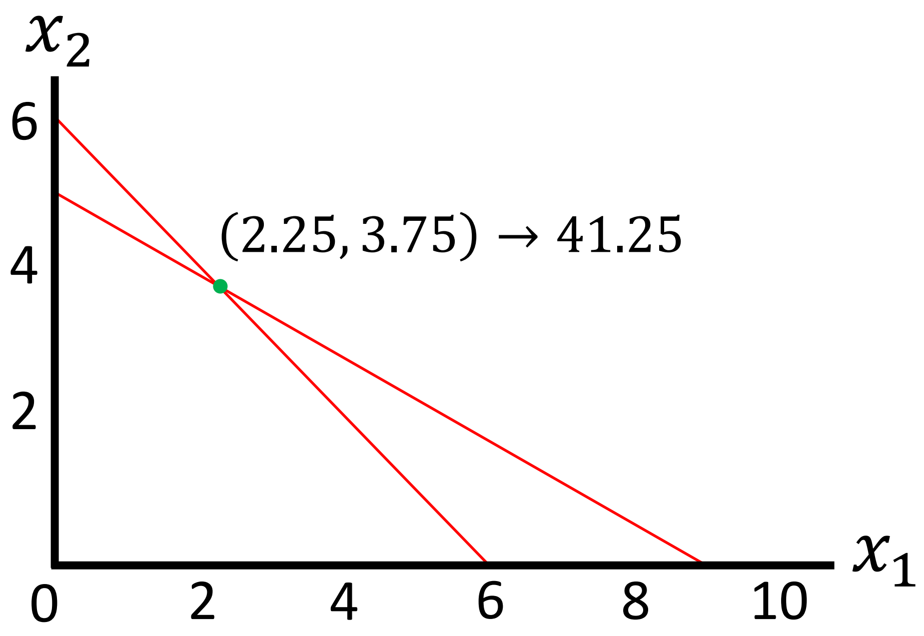


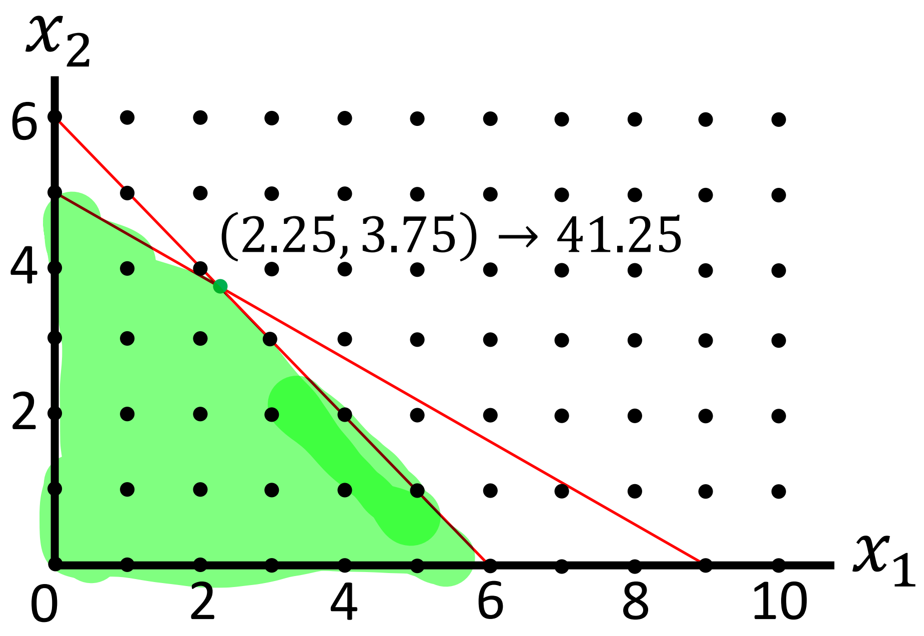
$x_1, x_2 \in \mathbb{N}$ → natural # ints ≥0

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$



$x_1, x_2 \in \mathbb{N}$

Objective:	$\max 5x_1 + 8x_2$
Subject to:	$x_1 + x_2 \leq 6$
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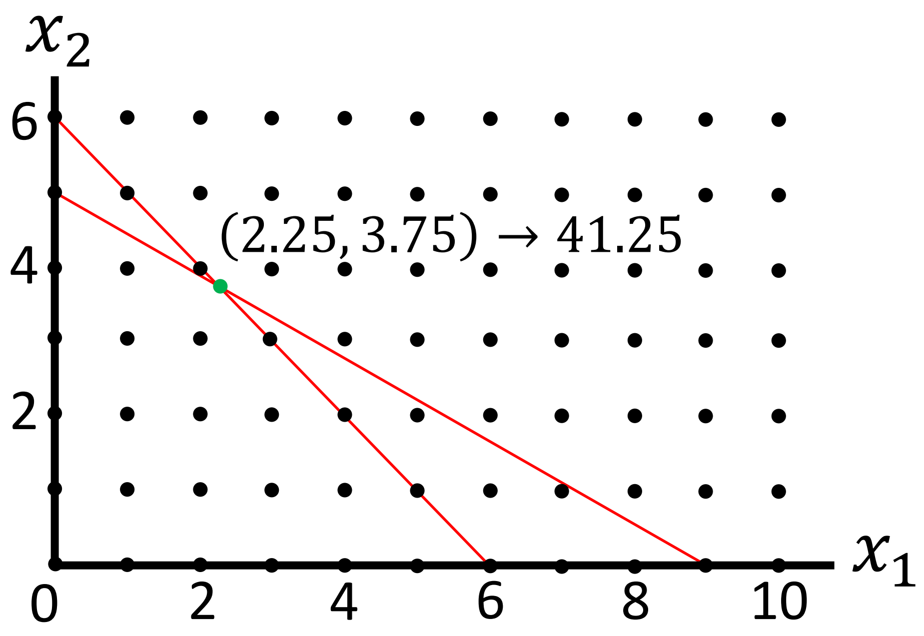


$x_1, x_2 \in \mathbb{N}$

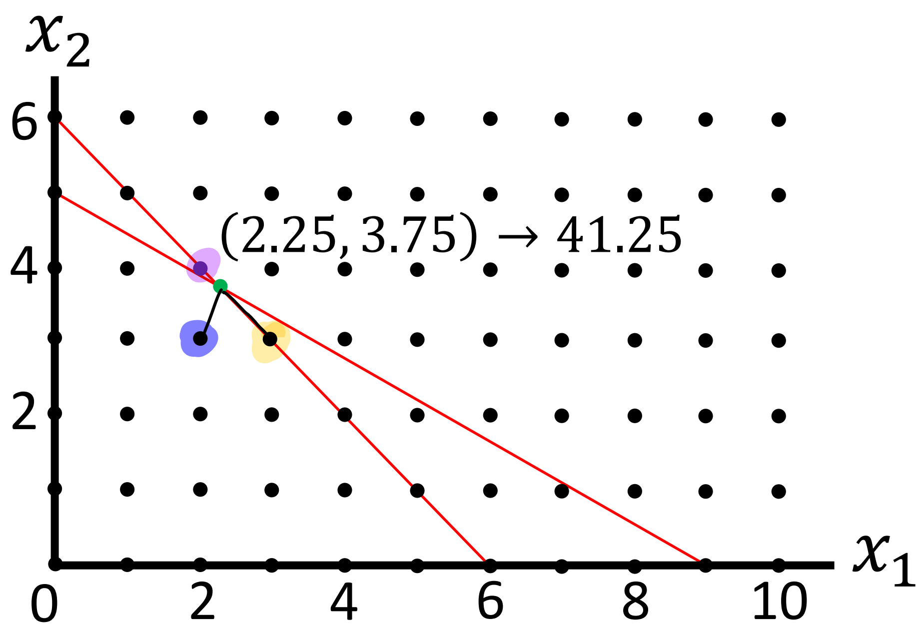
Objective: $\max 5x_1 + 8x_2$

Subject to:

$$x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, x_2 \geq 0$$

Optimal continuous solution → optimal integer solution?

- Closest integer solution?
- Closest feasible integer solution?
- Closest feasible integer solution on feasible region boundary?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

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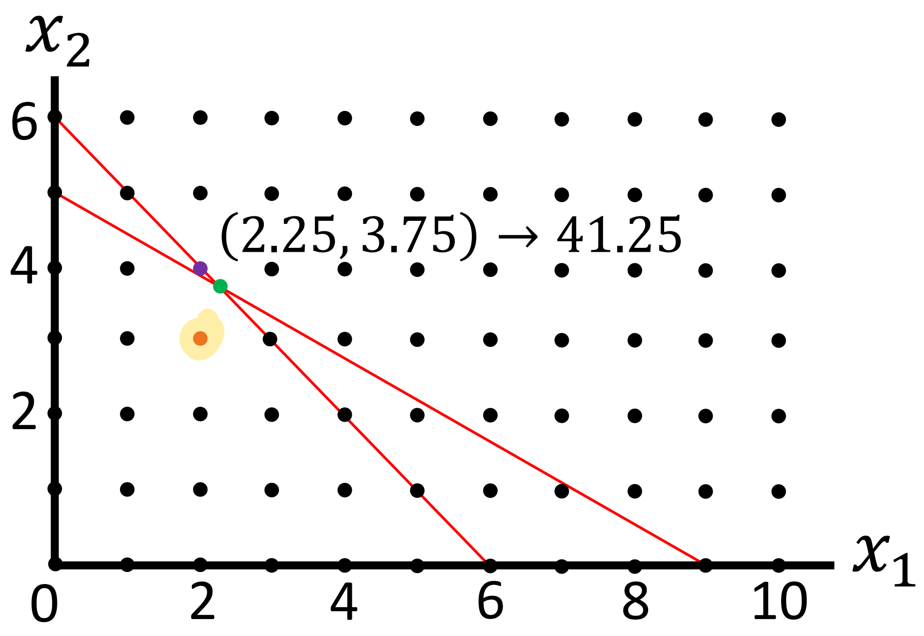
Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution?
- Closest feasible integer solution on feasible region boundary?



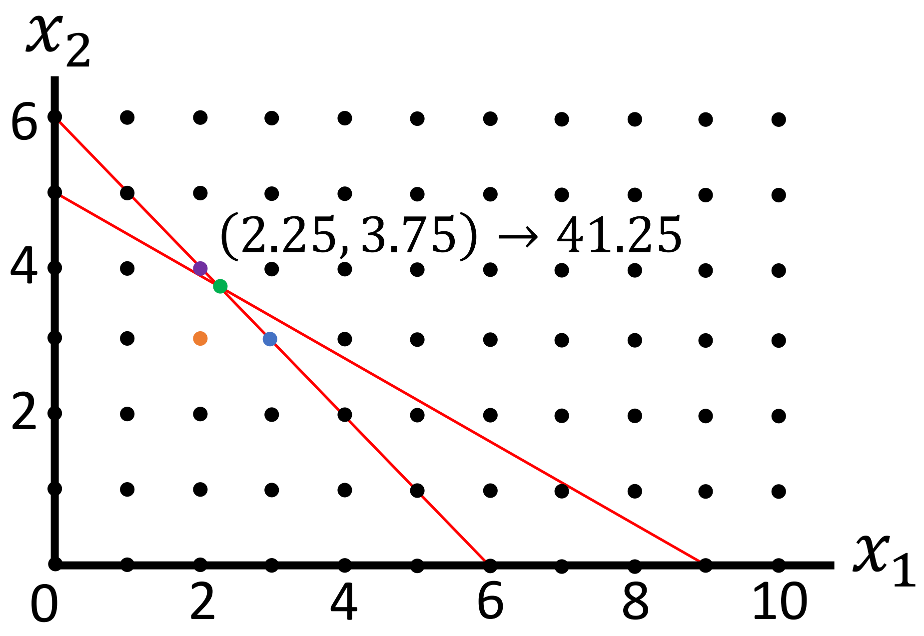
$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to:
$$x_1 + x_2 \le 6$$
$$5x_1 + 9x_2 \le 45$$
$$x_1, x_2 \ge 0$$

Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary?



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Objective: $\max 5x_1 + 8x_2$

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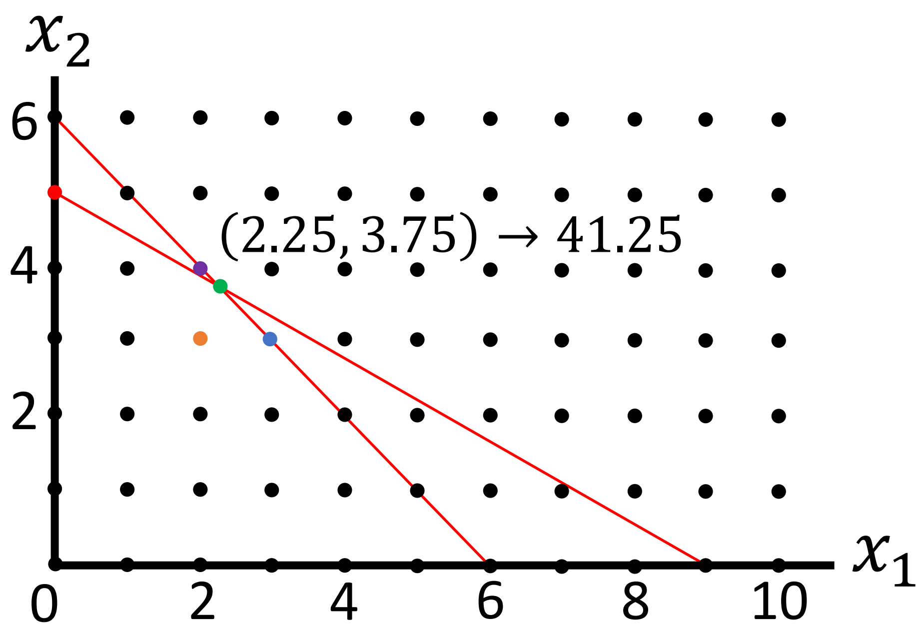
Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39



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Objective: $\max 5x_1 + 8x_2$

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$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39
- **Actual optimal – Obj = 40**

draw the "feasible region" for

$x_1, x_2 \in \mathbb{N}$

~~Objective:~~ $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

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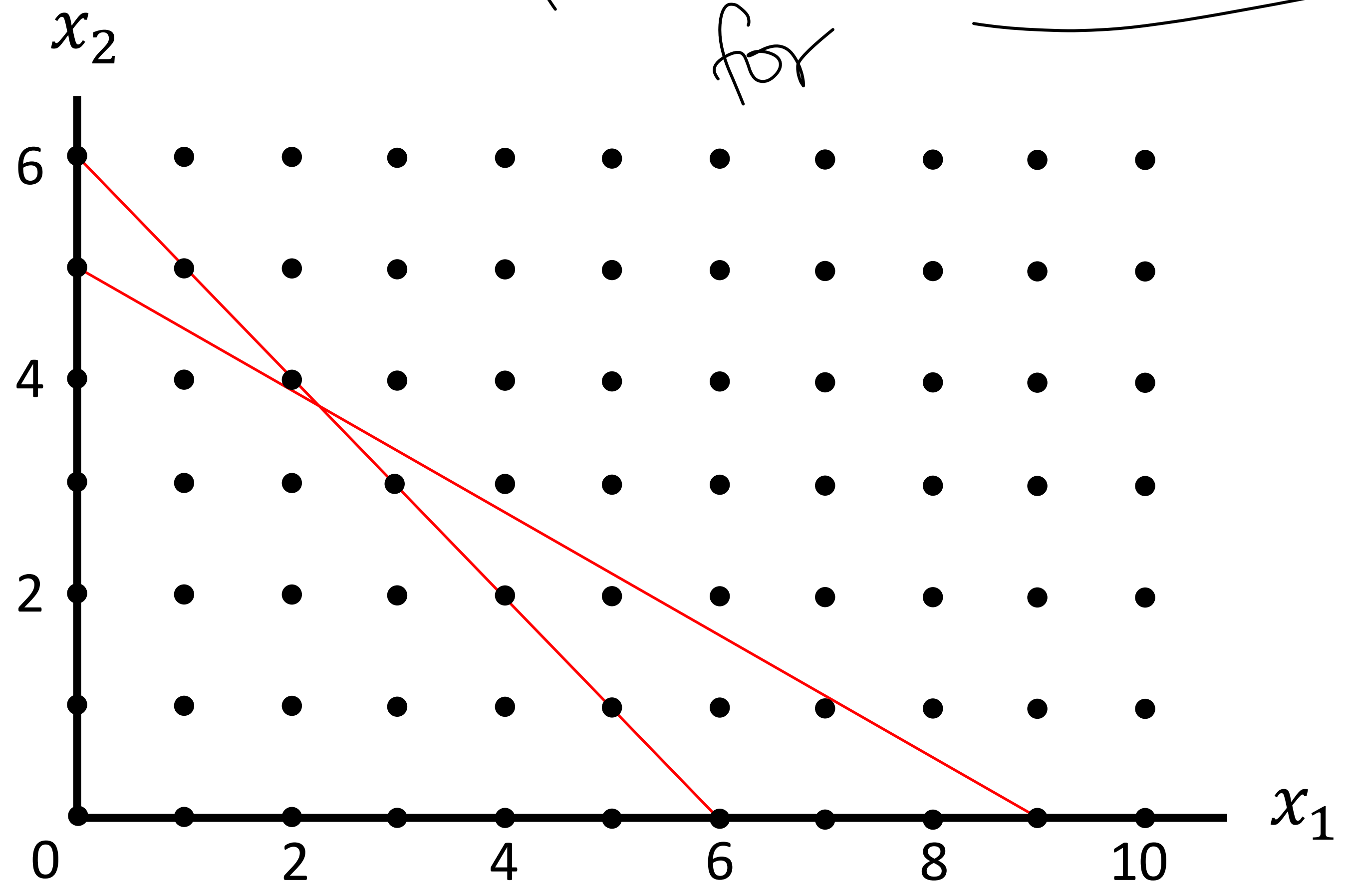

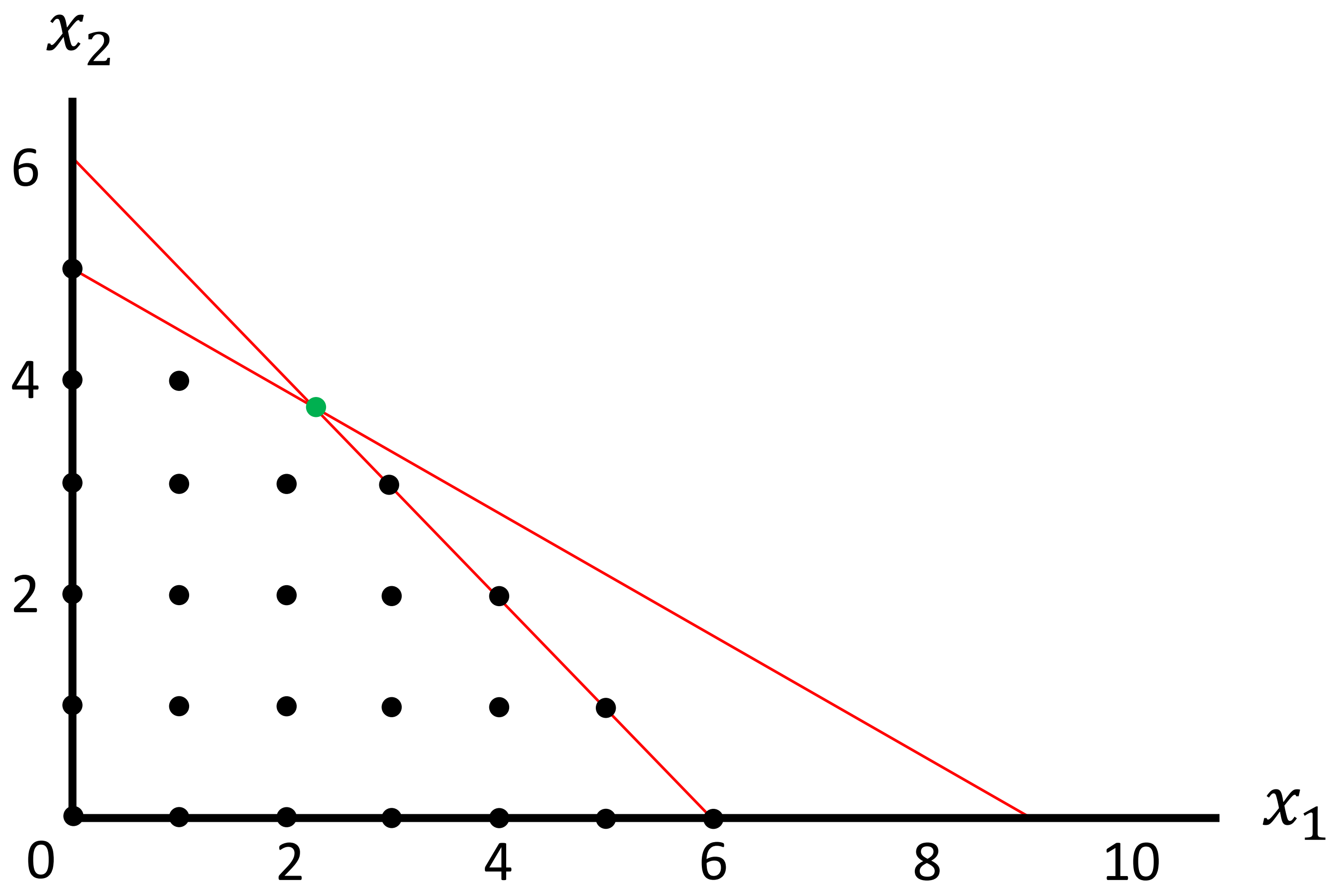

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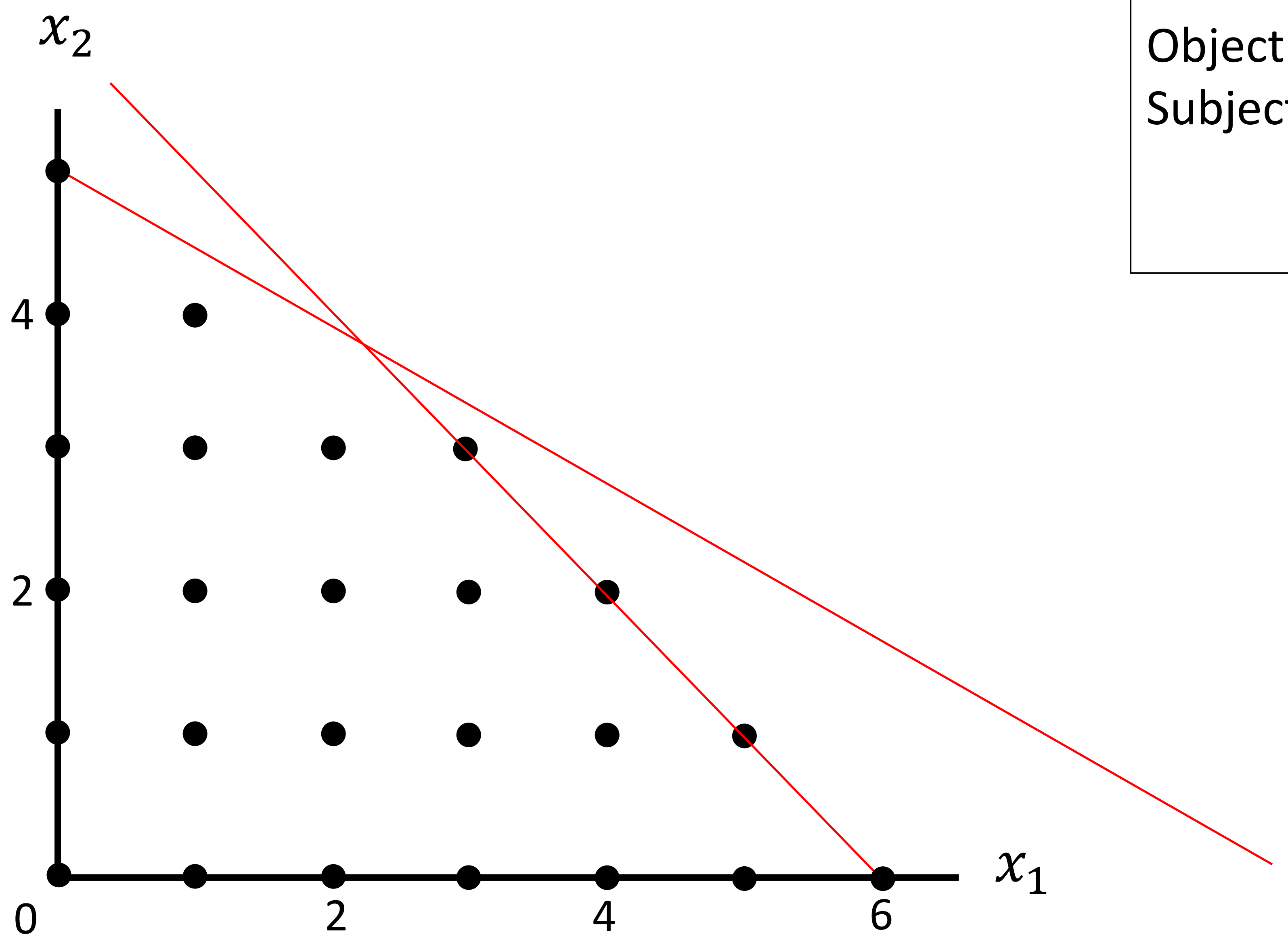

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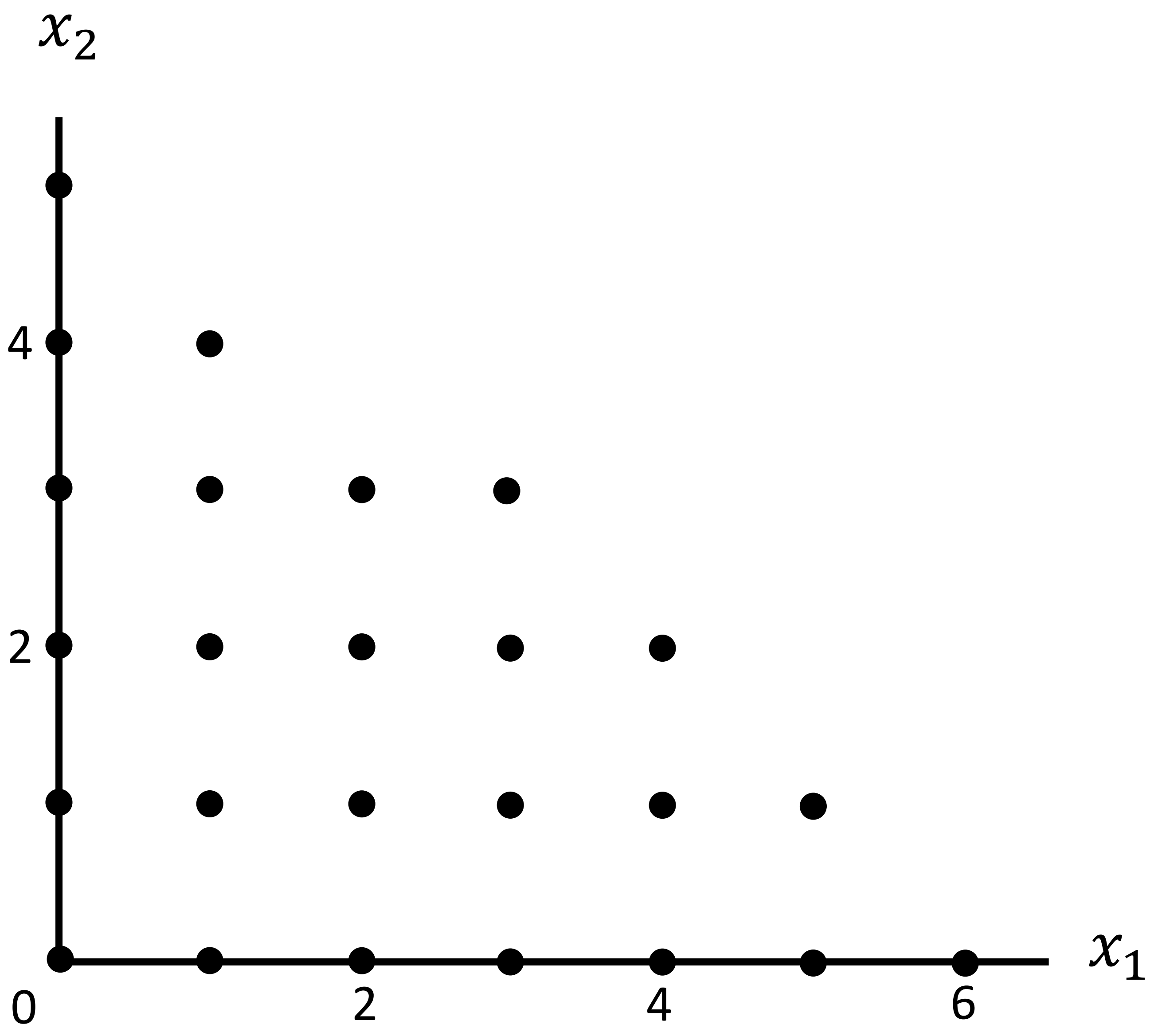

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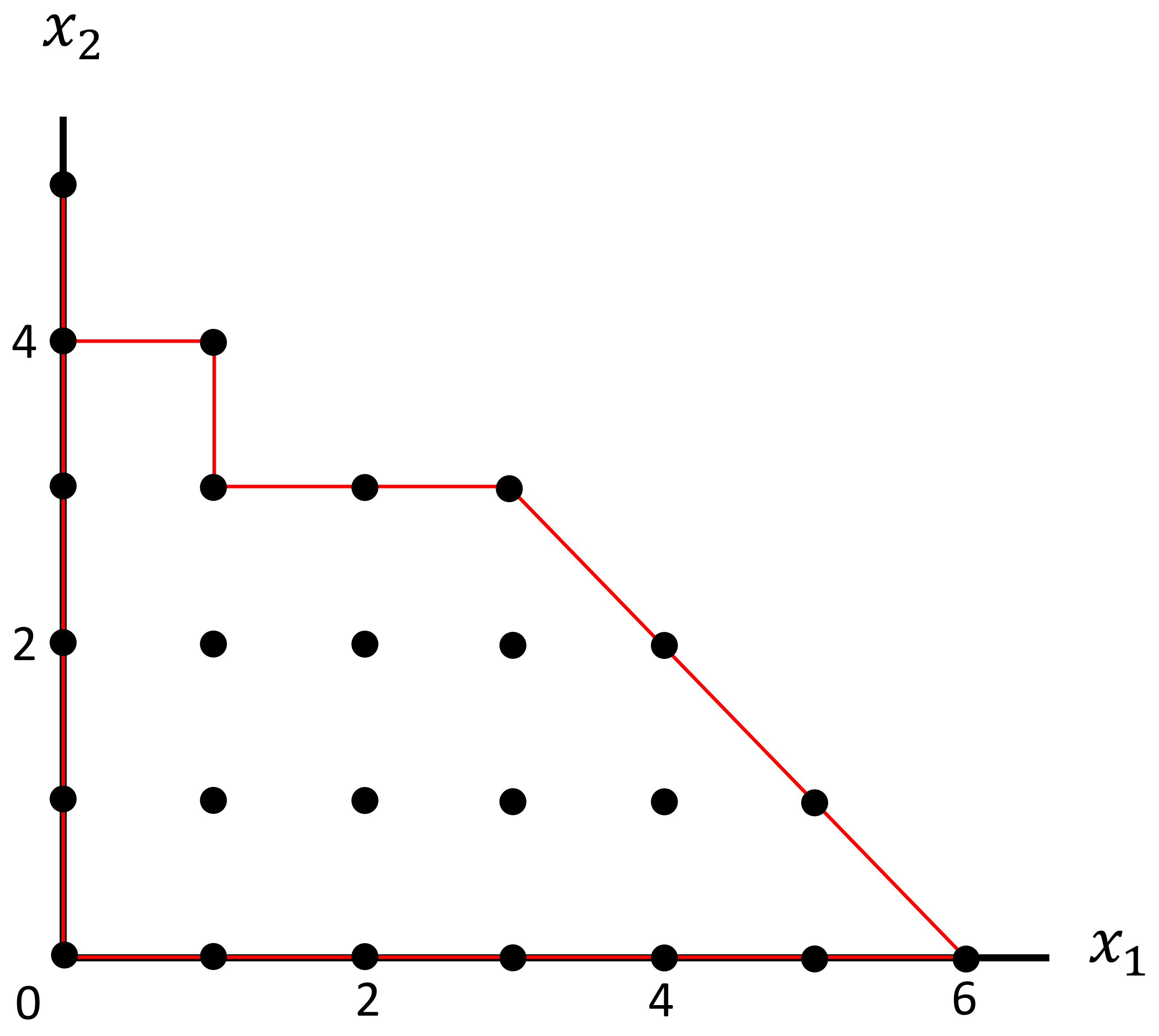
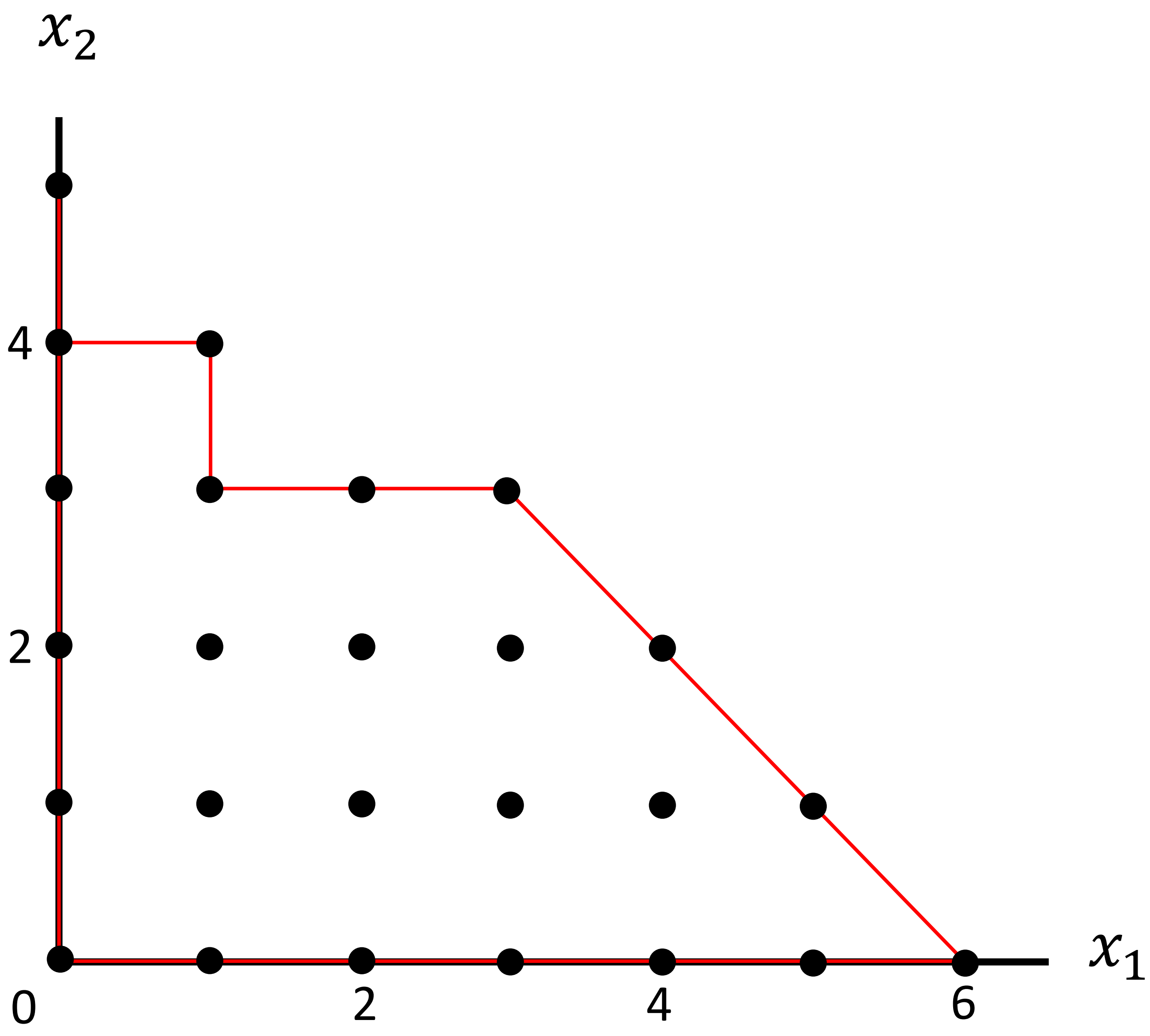
Integer feasible region:





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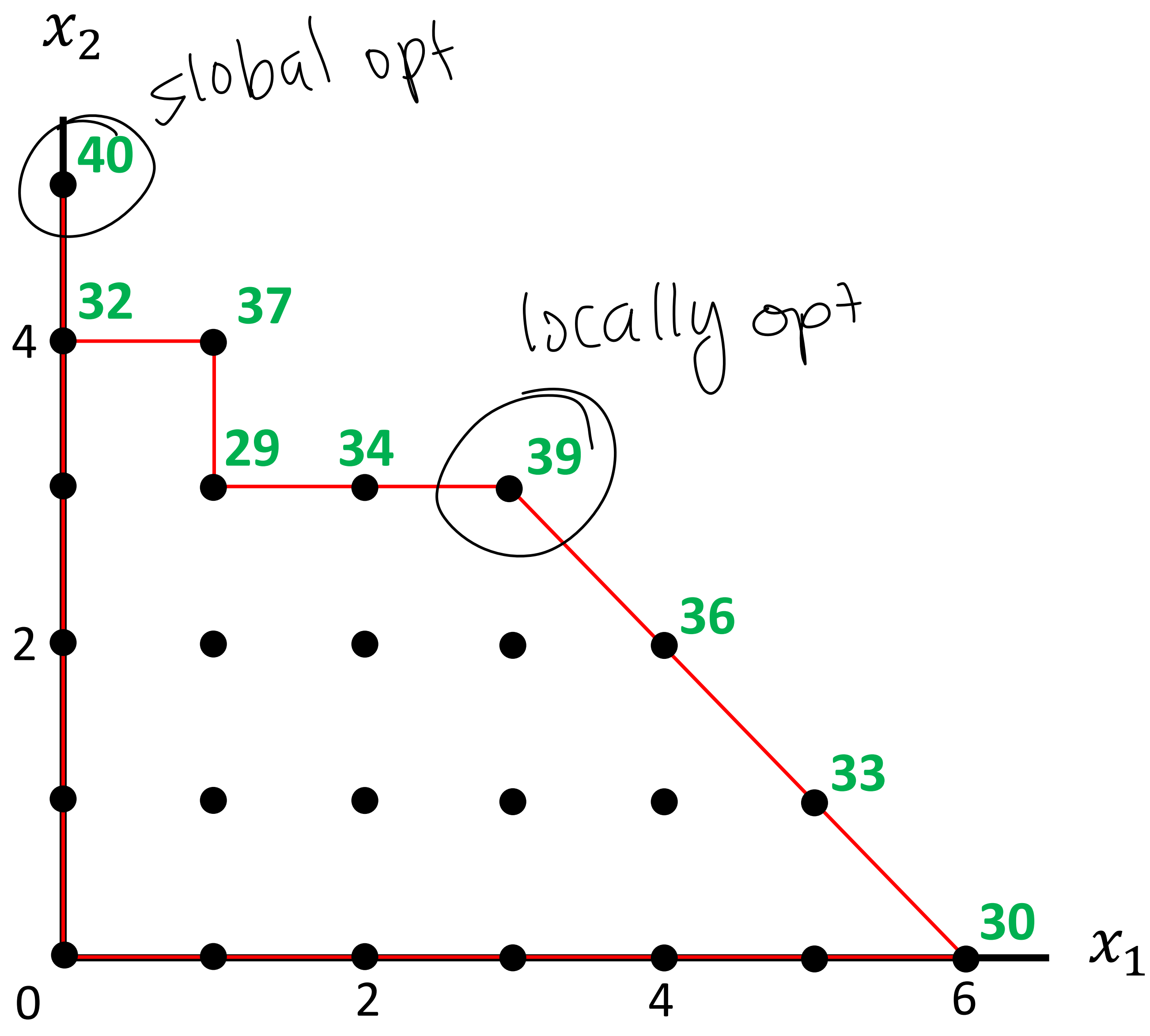
Integer feasible region:

- Not convex.



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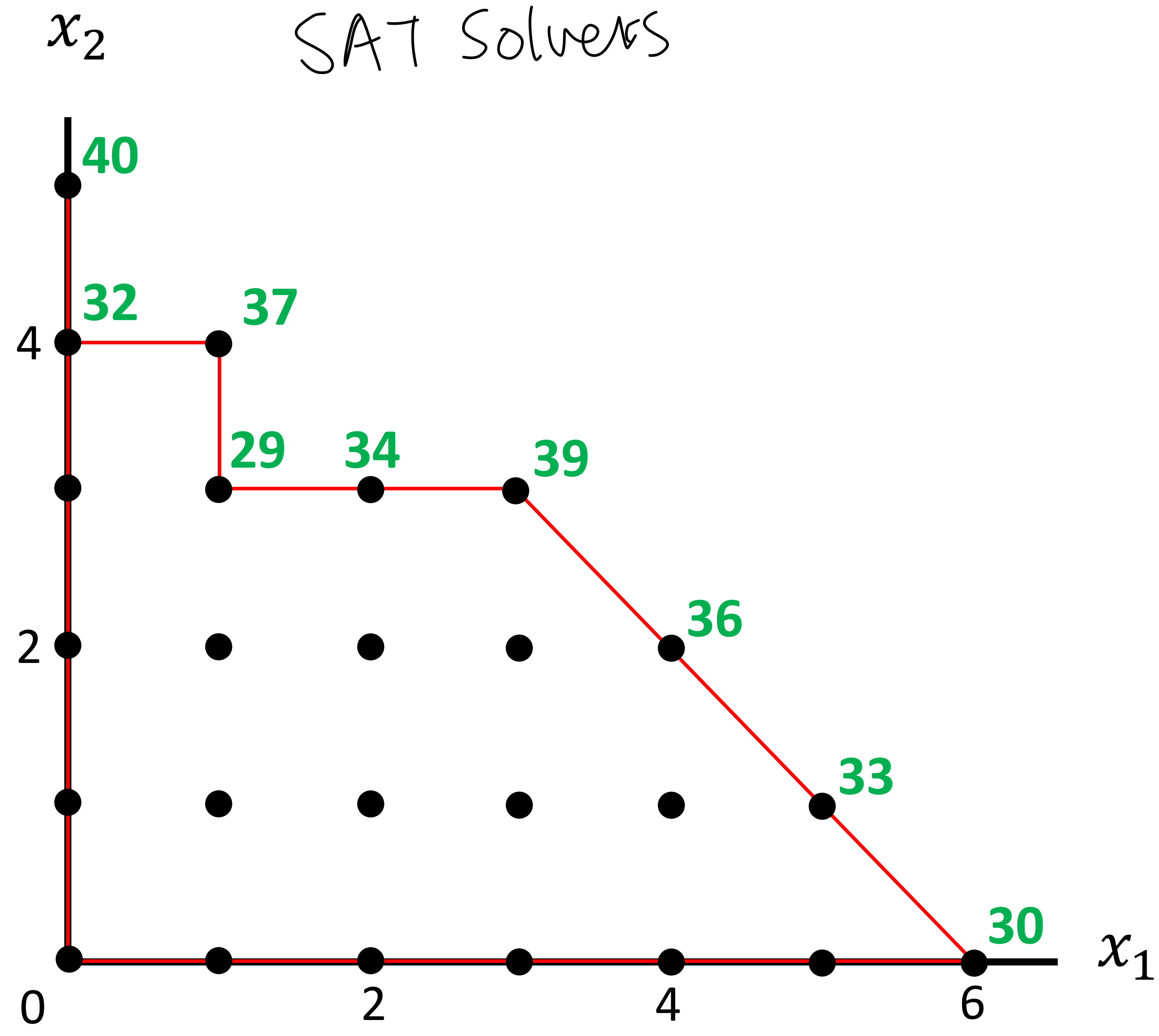
Integer feasible region:

- Not convex.

ILP Solvers
SAT Solvers



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Subject to:	$x_1 + x_2 \leq 6$
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	$x_1, x_2 \geq 0$

Integer feasible region:

- Not convex.
- local optimum $\neq$ global optimum.

Vertex Cover ILP – approximation

$ALG \leq \alpha \, OPT$

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$x_i \in \{0,1\}$, for each vertex i

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$\boldsymbol{x_i \in \{0, 1\}}$, **for each vertex** $\boldsymbol{i}$

$\in$ **NP-Hard**

Vertex Cover ILP

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∈ NP-Hard

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$\boldsymbol{0 \leq x_i \leq 1}$**, for each vertex** $\boldsymbol{i}$

∈ P

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$\boldsymbol{x_i \in \{0, 1\}}$**, for each vertex** $\boldsymbol{i}$

$\in$ **NP-Hard**

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$\boldsymbol{0 \leq x_i \leq 1}$**, for each vertex** $\boldsymbol{i}$

$\in$ **P**

LP Relaxation: Remove all integrality constraints to turn ILP into LP.

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

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Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

? ⇒ Vertex Selection

Vertex Cover ILP

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Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
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? → Vertex Selection

If $x_i = 1$, what should we do with vertex i?

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: $\min \sum_i x_i$
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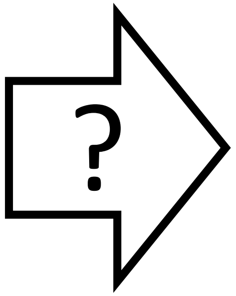
$0 \leq x_i \leq 1$, for each vertex i



Vertex Selection

If $x_i = 1$, what should we do with vertex i? Add to subset S
If $x_i = 0$, what should we do with vertex i?

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective: $\min \sum_i x_i$
Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$
$0 \leq x_i \leq 1$, for each vertex i

? → Vertex Selection

If $x_i = 1$, what should we do with vertex i? Add to subset S
If $x_i = 0$, what should we do with vertex i? Don't add to subset S

Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

? ⇒ Vertex Selection

If $x_i = 1$, what should we do with vertex i? Add to subset S

If $x_i = 0$, what should we do with vertex i? Don’t add to subset S

If $x_i = \frac{126}{337}$, what should we do with vertex i?

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+

If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+ If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Is S a vertex cover?

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
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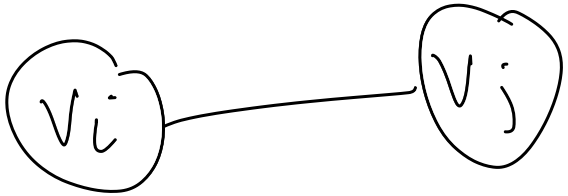

Is S a vertex cover?

Yes. For every edge, $x_i + x_j \geq 1$.

$x_i = 0$?

$x_j = 0$

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+

If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Is S a vertex cover?

Yes. For every edge, $x_i + x_j \geq 1$. Thus, at least one of x_i or $x_j \geq \frac{1}{2}$.

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+ If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Is S a vertex cover?

Yes. For every edge, $x_i + x_j \geq 1$. Thus, at least one of x_i or $x_j \geq \frac{1}{2}$. So for every edge, at least one of its vertices will be in S.

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+

If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

What is the relationship between $\text{ALG} = |S|$ and OPT?

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
	$0 \leq x_i \leq 1$, for each vertex i

+

If $x_i \geq \frac{1}{2}$, add vertex i to our subset S.

Can we bound OPT from below?

Vertex Cover ILP

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$
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Claim: $\sum x_{LP} \leq$ OPT.

Proof: OPT $=$?

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Can we bound OPT from below?

Let x_{ILP} and x_{LP} be the set of x values found by the ILP and LP

$x_1 = 0.1$, $x_2 = 0.61$ ...

Claim: $\sum x_{LP} \leq$ OPT.

$x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ ...

Proof: OPT $= \sum x_{ILP}$, where $x_i \in \{0,1\}$...?

$\sum x_{LP} \leq \sum x_{ILP}$

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Can we bound OPT from below?

Let x_{ILP} and x_{LP} be the set of x values found by the ILP and LP

Claim: $\sum x_{LP} \leq$ OPT.

Proof: OPT $= \sum x_{ILP}$, where $x_i \in \{0,1\}$. When x_i is relaxed so that $0 \leq x_i \leq 1$, this gives more possibilities to further decrease $\sum_i x_i$. Thus, $\sum x_{LP} \leq$ OPT.

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Can we bound OPT from below?

Let x_{ILP} … d LP

Claim:

Proof: … d so

that 0 …

decrease $\sum_i x_i$. Thus, $\sum x_{LP} \leq$ OPT.

Law of LP Relaxations:

$OPT_{LP} \leq OPT_{ILP}$

(minimization problem)

Vertex Cover ILP

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How does $\sum \mathrm{x_{LP}}$ relate to ALG?

$$\sum \mathrm{x_{LP}} = \sum_{x_i \in \mathrm{x_{LP}}} x_i \geq \sum_{x_i \in \mathrm{x_{LP}} : x_i \geq \frac{1}{2}} x_i, \text{ because...?}$$

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How does $\sum x_{LP}$ relate to ALG?

0.2 + 0.6 + 0.9

$\sum x_{LP} = \sum_{x_i \in x_{LP}} x_i \geq \sum_{x_i \in x_{LP} : x_i \geq \frac{1}{2}} x_i$, because it's a subset of x_{LP}

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$$\geq \sum_{x_i \in \mathrm{x}_{\mathrm{LP}} : x_i \geq \frac{1}{2}} \frac{1}{2}$$, because...?

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$$\geq \sum_{x_i \in \mathrm{x}_{\mathrm{LP}} : x_i \geq \frac{1}{2}} \frac{1}{2}, \text{ because each } x_i \text{ is at least } \frac{1}{2}$$

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$$= \frac{1}{2} \left| \left\{ x_i \in \mathrm{x_{LP}}: x_i \geq \frac{1}{2} \right\} \right| = \frac{1}{2} \text{ALG}$$

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What is the relationship between ALG and OPT?

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$\sum x_{LP} \geq \frac{1}{2}$ ALG and $\sum x_{LP} \leq$ OPT

ALG ≤ 2 OPT

Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \left\{ \begin{matrix} \{1, 7, 8\}, \{1, 4, 7\}, \\ \{7, 8\}, \{4, 8, 10\} \end{matrix} \right\}$$

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ILP?

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$x_s \in \{0,1\}$, for each set s

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<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4$
Subject to:
$x_1 + x_2 \geq 1$
$x_2 + x_4 \geq 1$
$x_1 + x_2 + x_3 \geq 1$
$x_1 + x_3 + x_4 \geq 1$
$x_4 \geq 1$
$x_1, x_2, x_3, x_4 \in \{0,1\}$

$U = \{1, 4, 7, 8, 10\}$

$S = \left\{ \begin{matrix} \{1,7,8\}, \{1,4,7\}, \\ \{7,8\}, \{4,8,10\} \end{matrix} \right\}$

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$0 \leq x_s \leq 1$, for each set s

+

If $x_s \geq \frac{1}{2}$, add set s to our subset S_{ALG}.

Set Cover ILP

Objective:	$\min \sum_s x_s$
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If $x_s \geq \frac{1}{2}$, add set s to our subset S_{ALG}.

Could this lead to an invalid solution?

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If $x_s \geq \frac{1}{2}$, add set s to our subset S_{ALG}.

Could this lead to an invalid solution?

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 + x_3 \geq 1$
	$x_1 + x_2 + x_4 \geq 1$
	$x_1 + x_3 + x_4 \geq 1$
	$x_2 + x_3 + x_4 \geq 1$
	$x_1, x_2, x_3, x_4 \in [0,1]$

$$U = \{1, 2, 3, 4\}$$

$$S = \left\{ \begin{matrix} \{1,2,3\}, \{1,2,4\}, \\ \{1,3,4\}, \{2,3,4\} \end{matrix} \right\}$$

Set Cover ILP

Objective:	$\min \sum_s x_s$
Subject to:	$\sum_{s:\, u \in s} x_s \geq 1$, for each $u \in U$
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Subject to:	$x_1 + x_2 + x_3 \geq 1$
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	$x_2 + x_3 + x_4 \geq 1$
	$x_1, x_2, x_3, x_4 \in [0,1]$

$U = \{1, 2, 3, 4\}$

$S = \begin{Bmatrix} \{1,2,3\}, \{1,2,4\}, \\ \{1,3,4\}, \{2,3,4\} \end{Bmatrix}$

Yes, in this case $x_s = \frac{1}{3}, \forall s \Rightarrow$ No sets are selected (invalid solution).

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+ ?

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+

Add set s to our subset S_{ALG} with probability of x_s.