Vertex Cover Integer Linear Program

<u>Input</u>
Graph $G = (V, E)$
Where $V = \{v_1, v_2, \dots, v_n\}$
and $E = \{\{v_i, v_j\}$ where $v_i, v_j \in V\}$

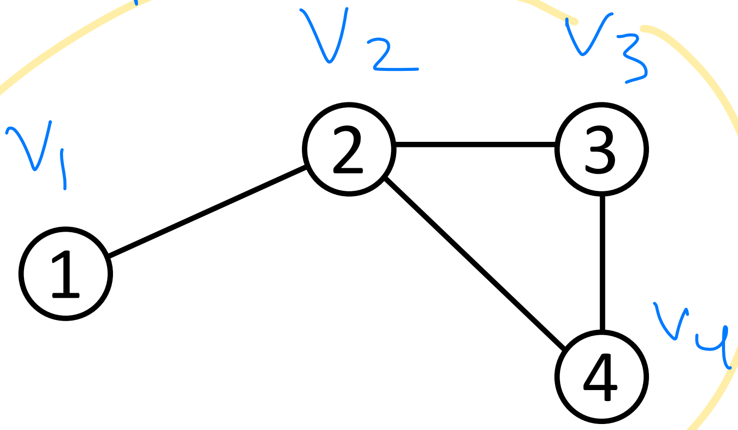
<u>Output</u>
smallest set of vertices covering all edges

<u>Linear Program</u>

vars: x_i for each vert v_i
$x_i \in \{0, 1\}$

objective:
$\min \sum x_i$

constraints:
for each edge, sum of endpoints ≥ 1



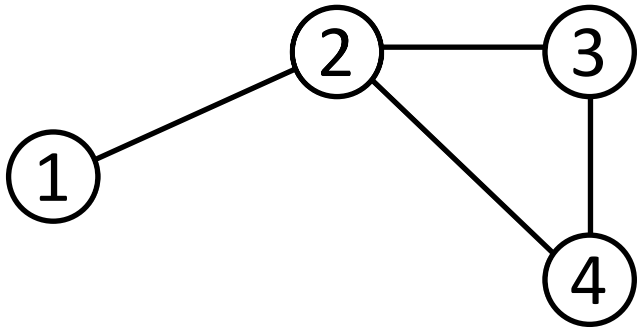
Vertex Cover ILP

Vertex Cover: Given graph, find the smallest subset of vertices such that every edge in the graph has at least one vertex in the subset.

Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $\{v_i, v_j\}$
	$x_i \in \{0,1\}$, for each vertex i

Example:

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 \geq 1$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



Vertex Cover ILP

integer

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Example:

Objective:	$\min x_1 + x_2 + x_3 + x_4$
Subject to:	$x_1 + x_2 \geq 1$
	$x_2 + x_3 \geq 1$
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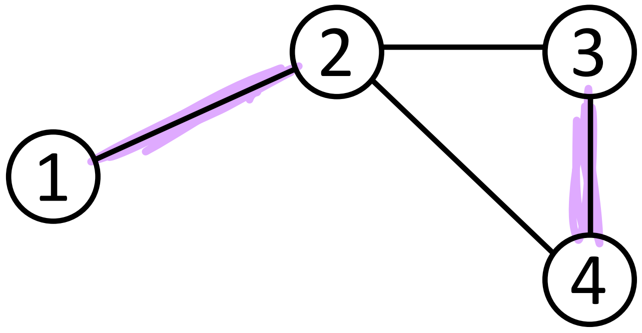
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

$$U = \{1, 4, 7, 8, 10\}$$

$$S = \left\{ \begin{array}{c} \{1, 7, 8\}, \{1, 4, 7\}, \\ \{7, 8\}, \{4, 8, 10\} \end{array} \right\}$$

Set Cover ILP

Set Cover: Given a universe of elements U and sets S, find the smallest subset of S such that every element in U is in some selected subset.

Objective:	$\min \sum_s x_s$
Subject to:	$\sum_{s:\, u \in s} x_s \geq 1$, for each $u \in U$
	$x_s \in \{0,1\}$, for each set s

$$U = \{1, 4, 7, 8, 10\}$$

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<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4$
Subject to:
$x_1 + x_2 \geq 1$
$x_2 + x_4 \geq 1$
$x_1 + x_2 + x_3 \geq 1$
$x_1 + x_3 + x_4 \geq 1$
$x_4 \geq 1$
$x_1, x_2, x_3, x_4 \in \{0,1\}$

$U = \{1, 4, 7, 8, 10\}$

$S = \left\{ \begin{matrix} \{1,7,8\}, \{1,4,7\}, \\ \{7,8\}, \{4,8,10\} \end{matrix} \right\}$

We now have a poly-time reduction from Vertex Cover to Set Cover

Integer linear programming

objective, constraints linear funcs of vars

LP

Vertex Cover and Set Cover are NP-hard

NP-hard = if we can solve in polynomial time, then P = NP

ILP is NP-hard

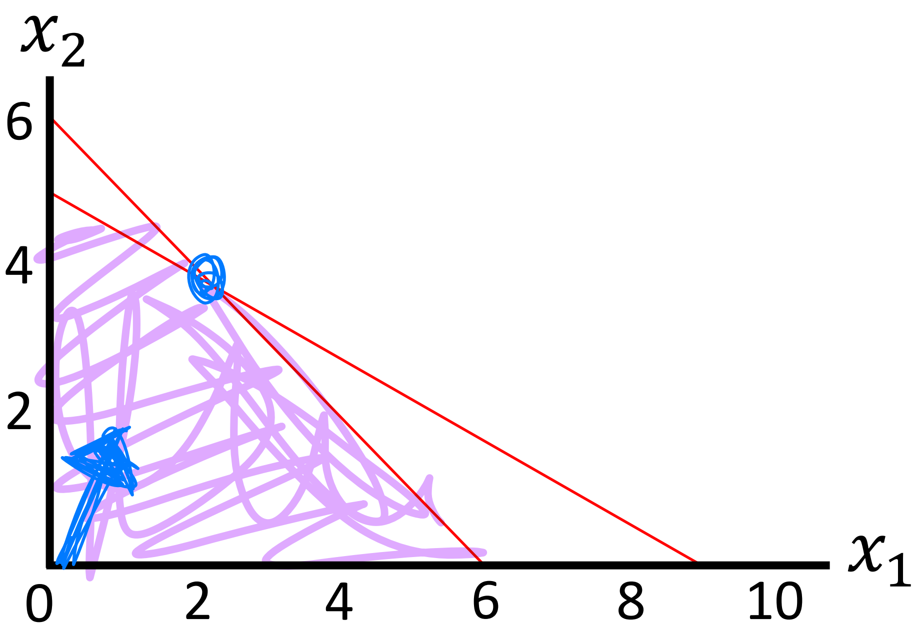
$x_1, x_2 \in \mathbb{R}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$



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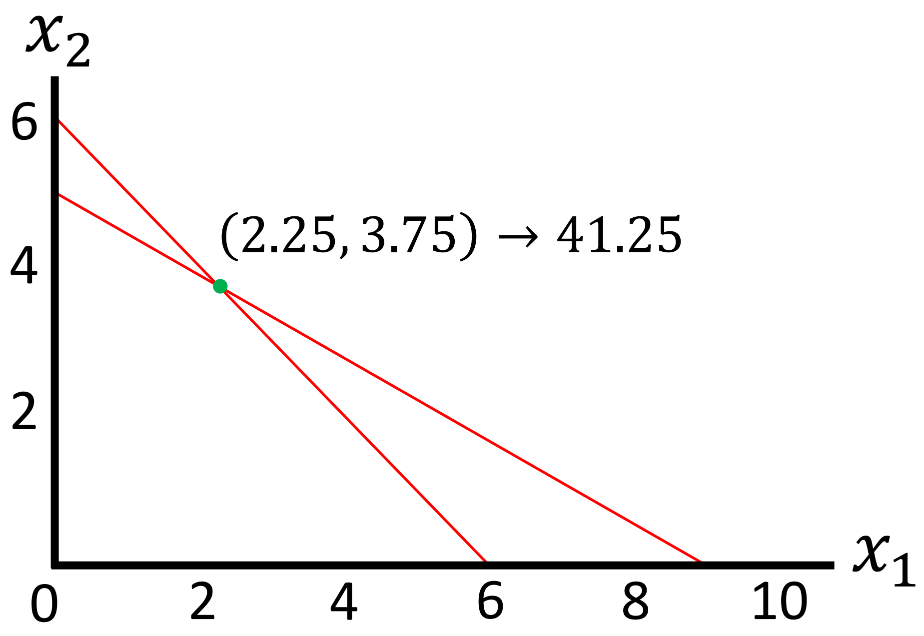

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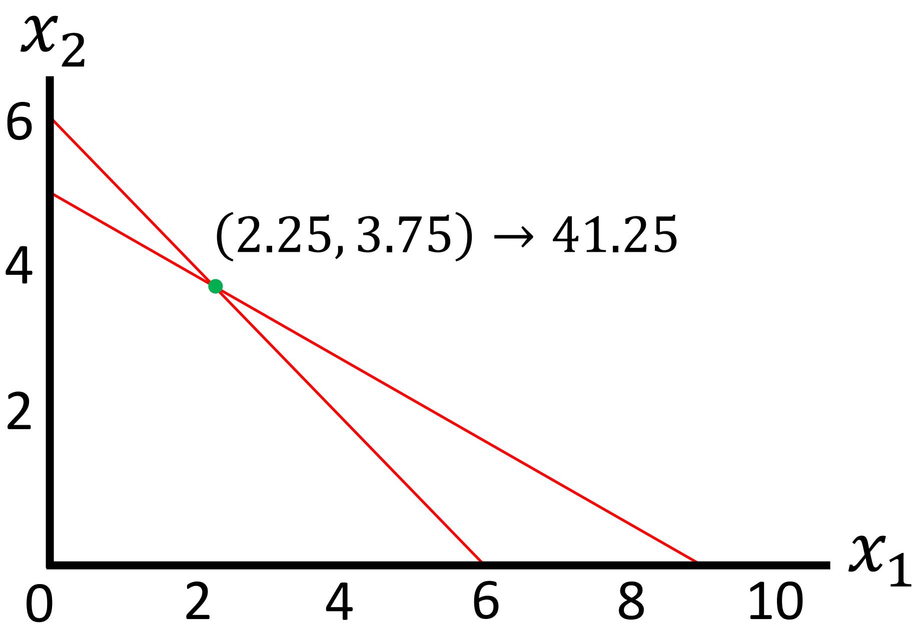

$x_1, x_2 \in \mathbb{N}$ integers ≥ 0

Objective:	$\max 5x_1 + 8x_2$
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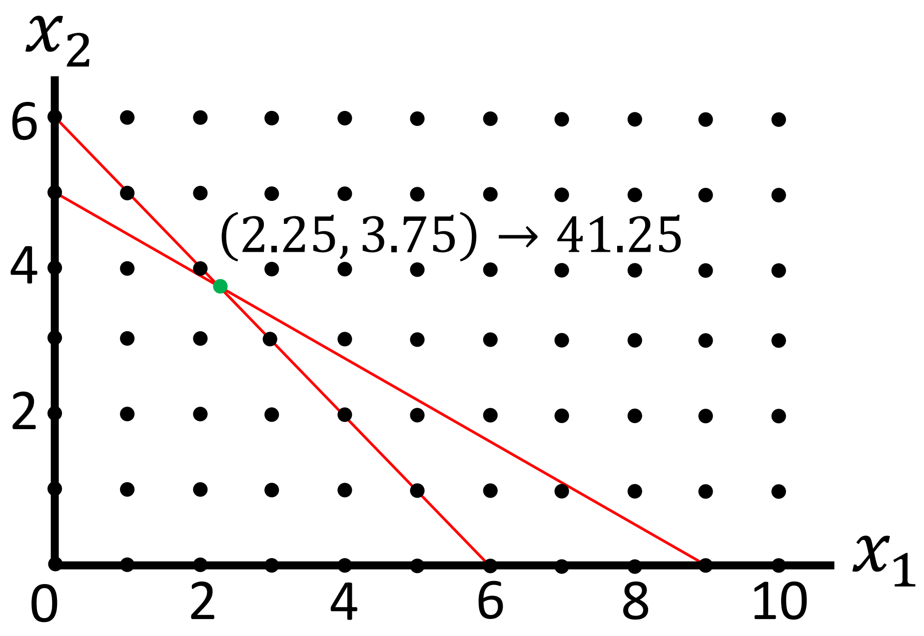

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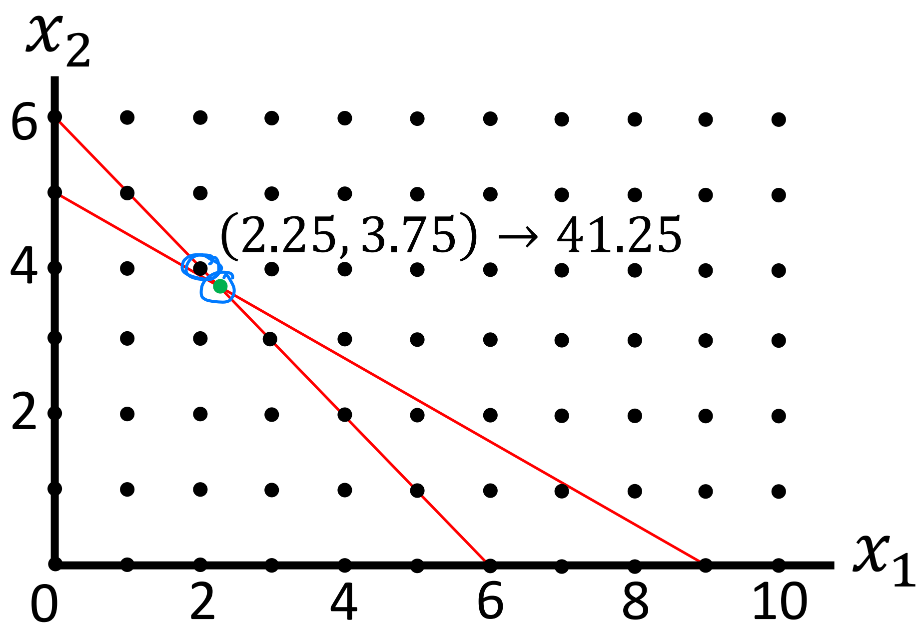

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Objective: $\max 5x_1 + 8x_2$

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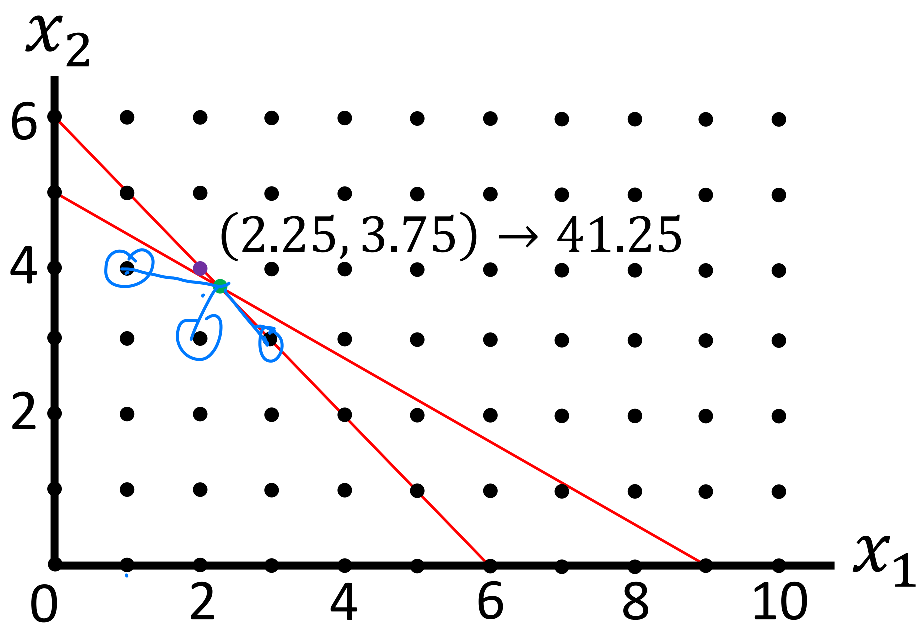
$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- Closest integer solution?
- Closest feasible integer solution?
- Closest feasible integer solution on feasible region boundary?



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Optimal continuous solution → optimal integer solution?

- Closest integer solution? – Not feasible
- Closest feasible integer solution?
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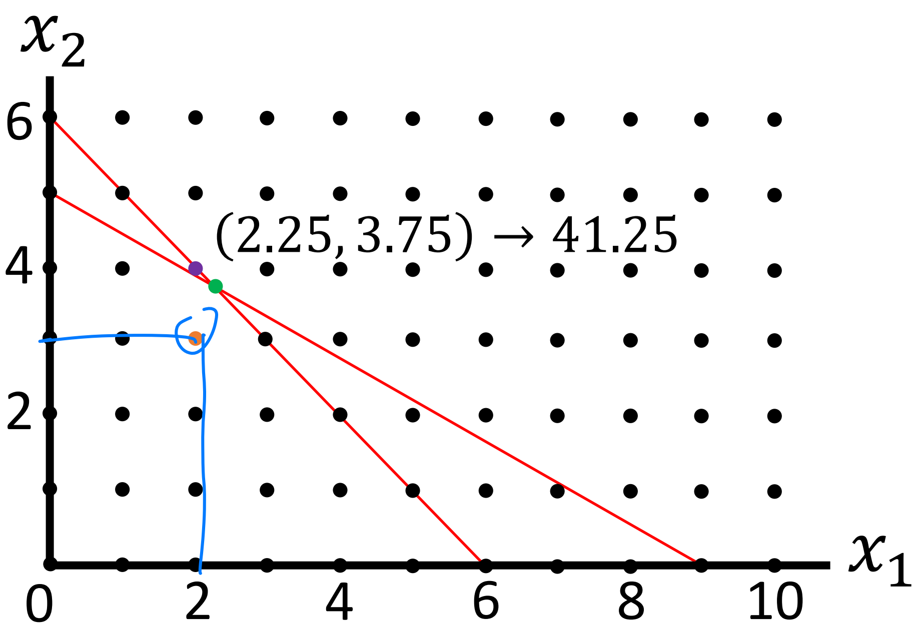

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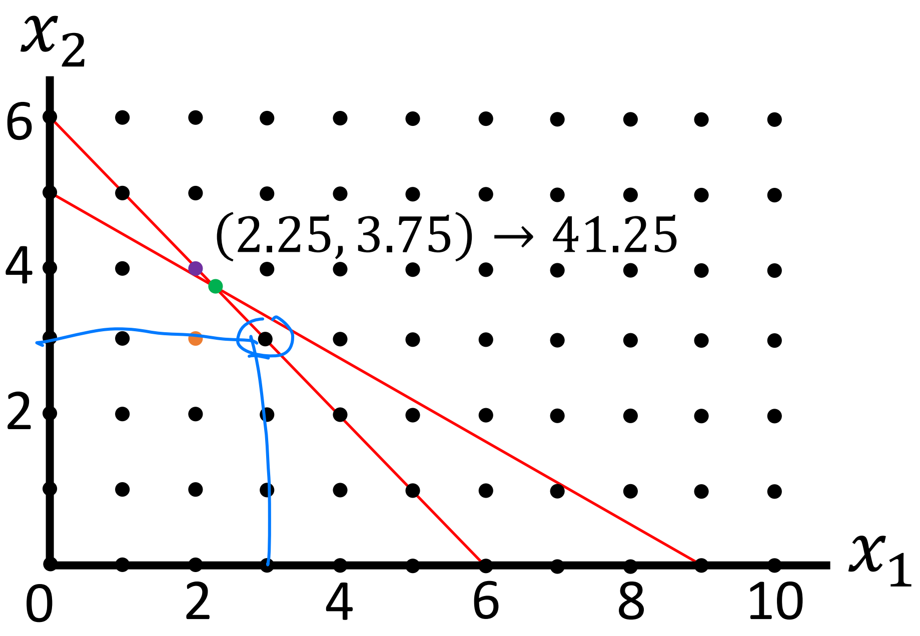

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Objective: $\max 5x_1 + 8x_2$

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Optimal continuous solution → optimal integer solution?

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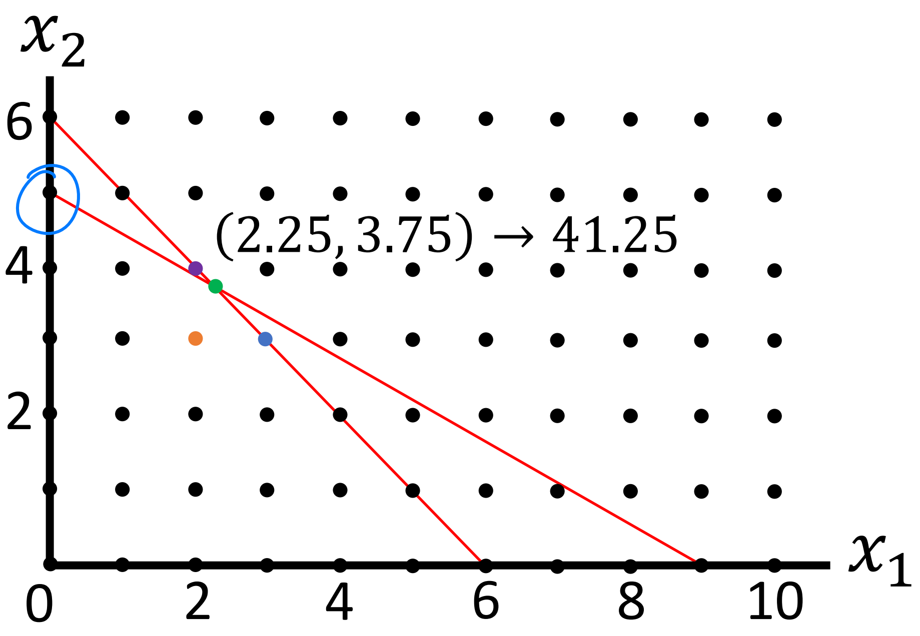
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary?



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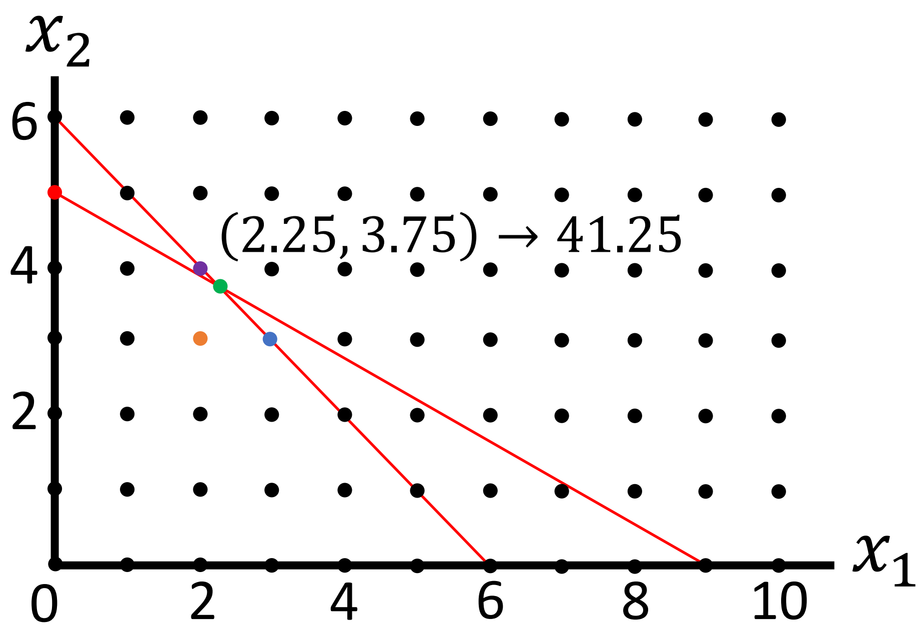
- Closest feasible integer solution? – Obj = 34
- Closest feasible integer solution on feasible region boundary? – Obj = 39



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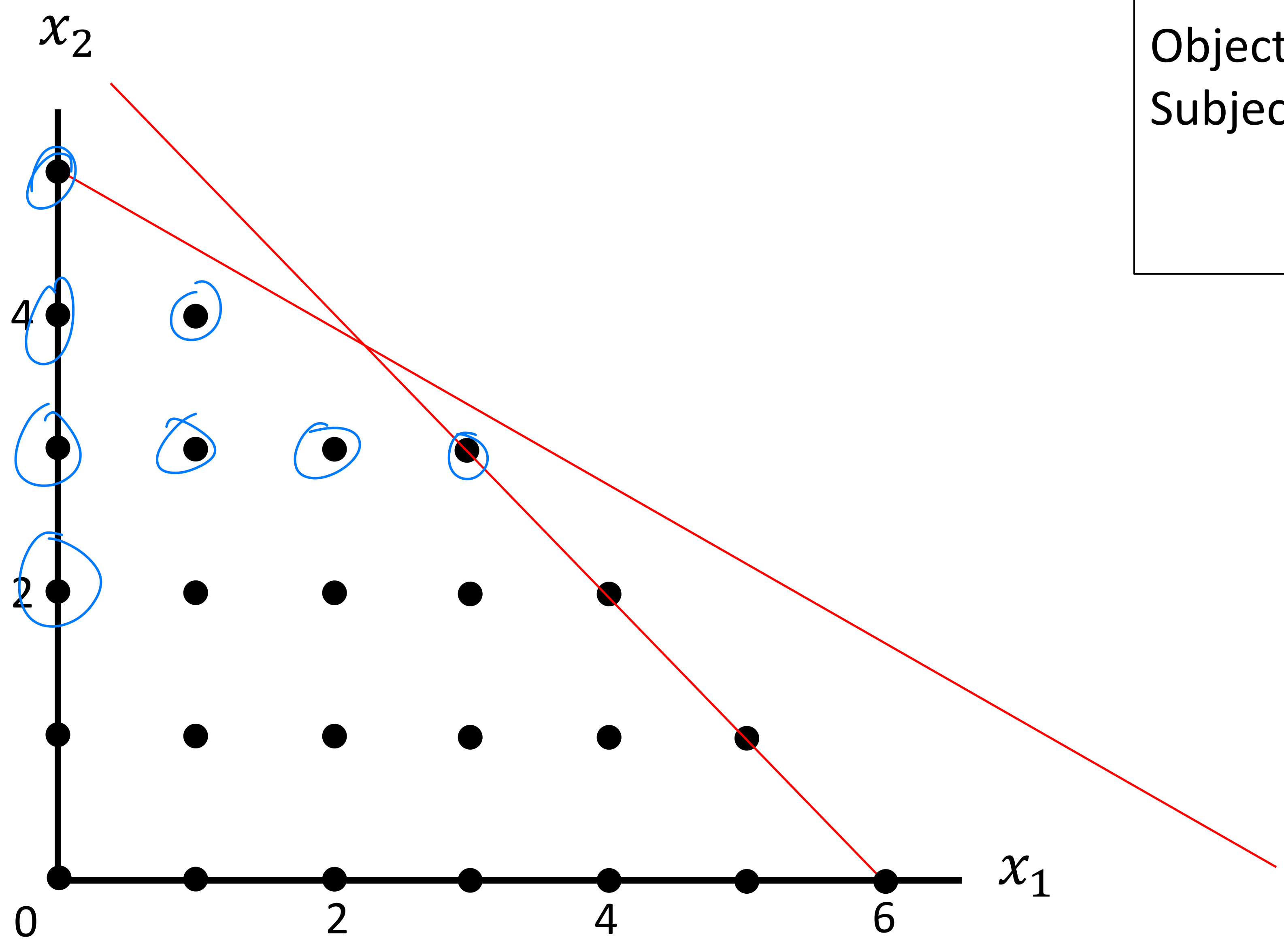
- **Actual optimal – Obj = 40**



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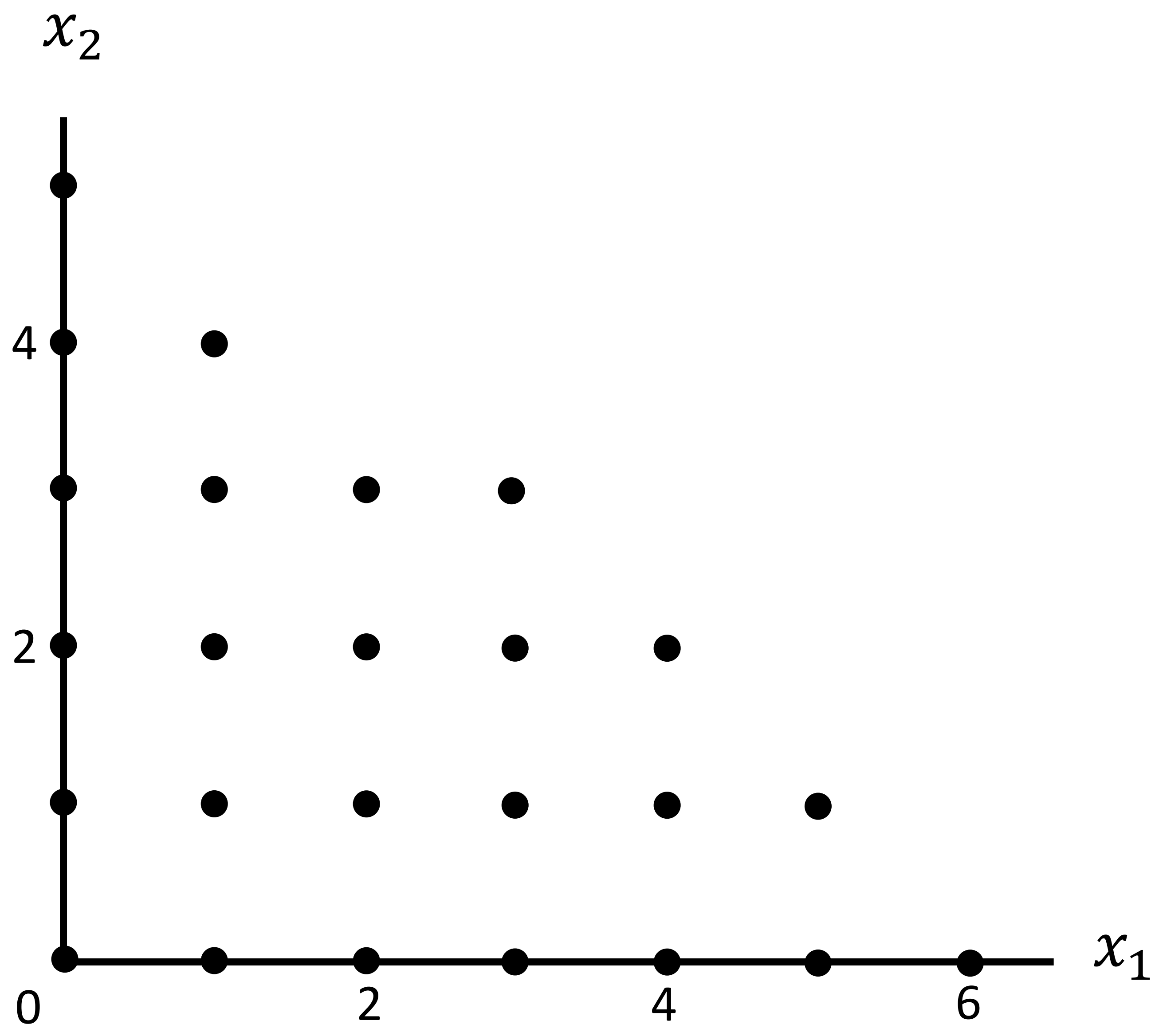

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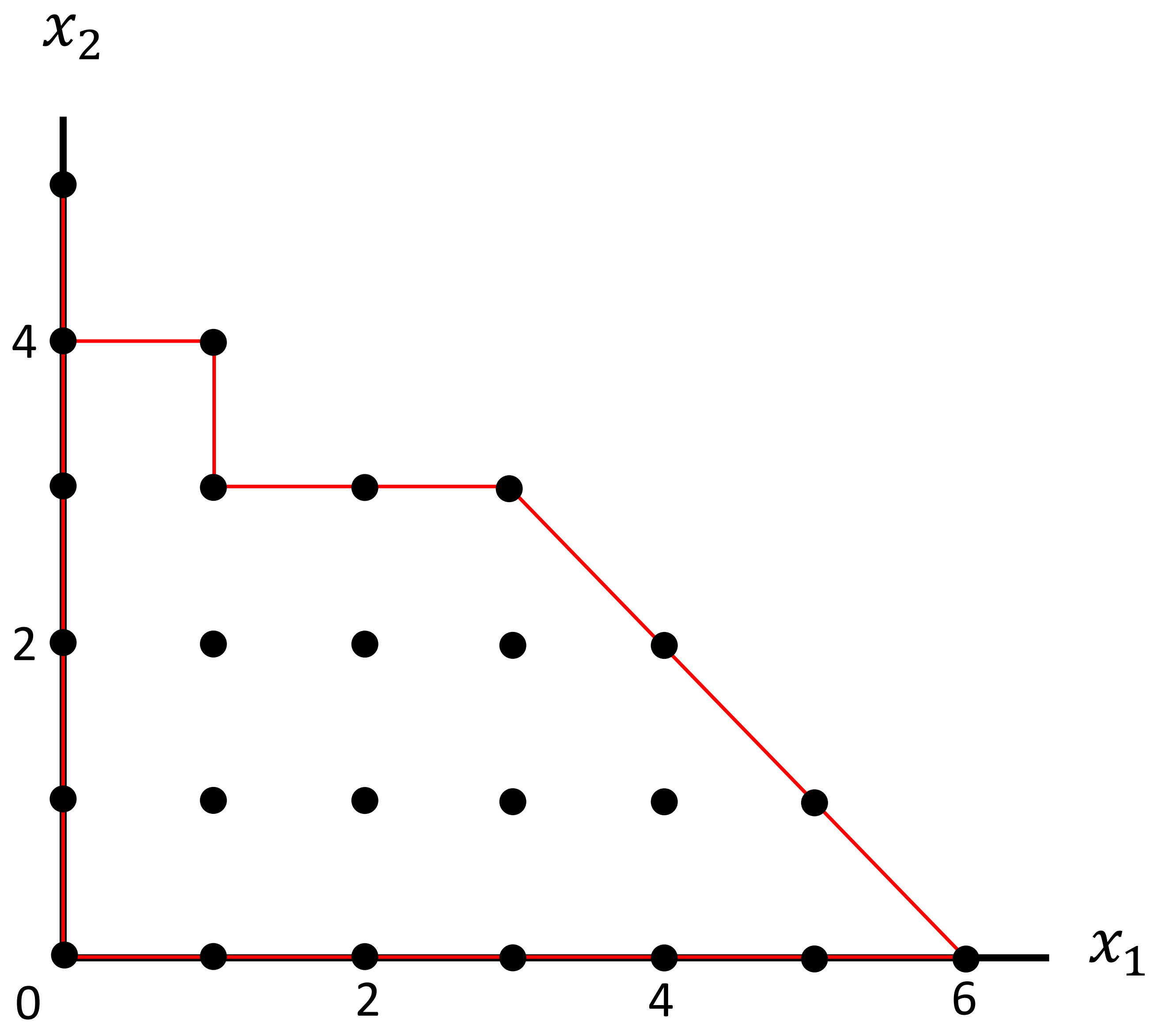

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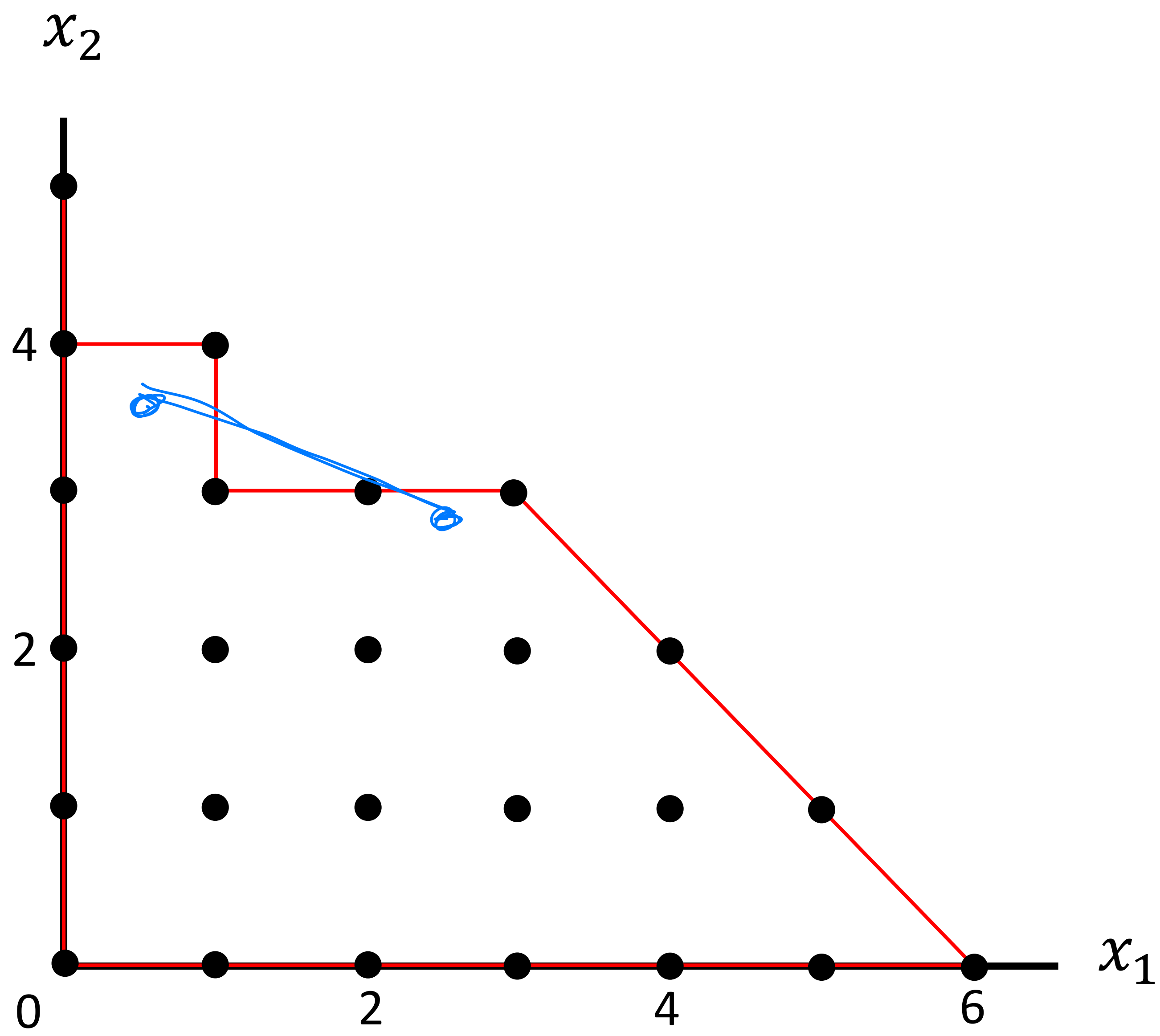
Integer feasible region:



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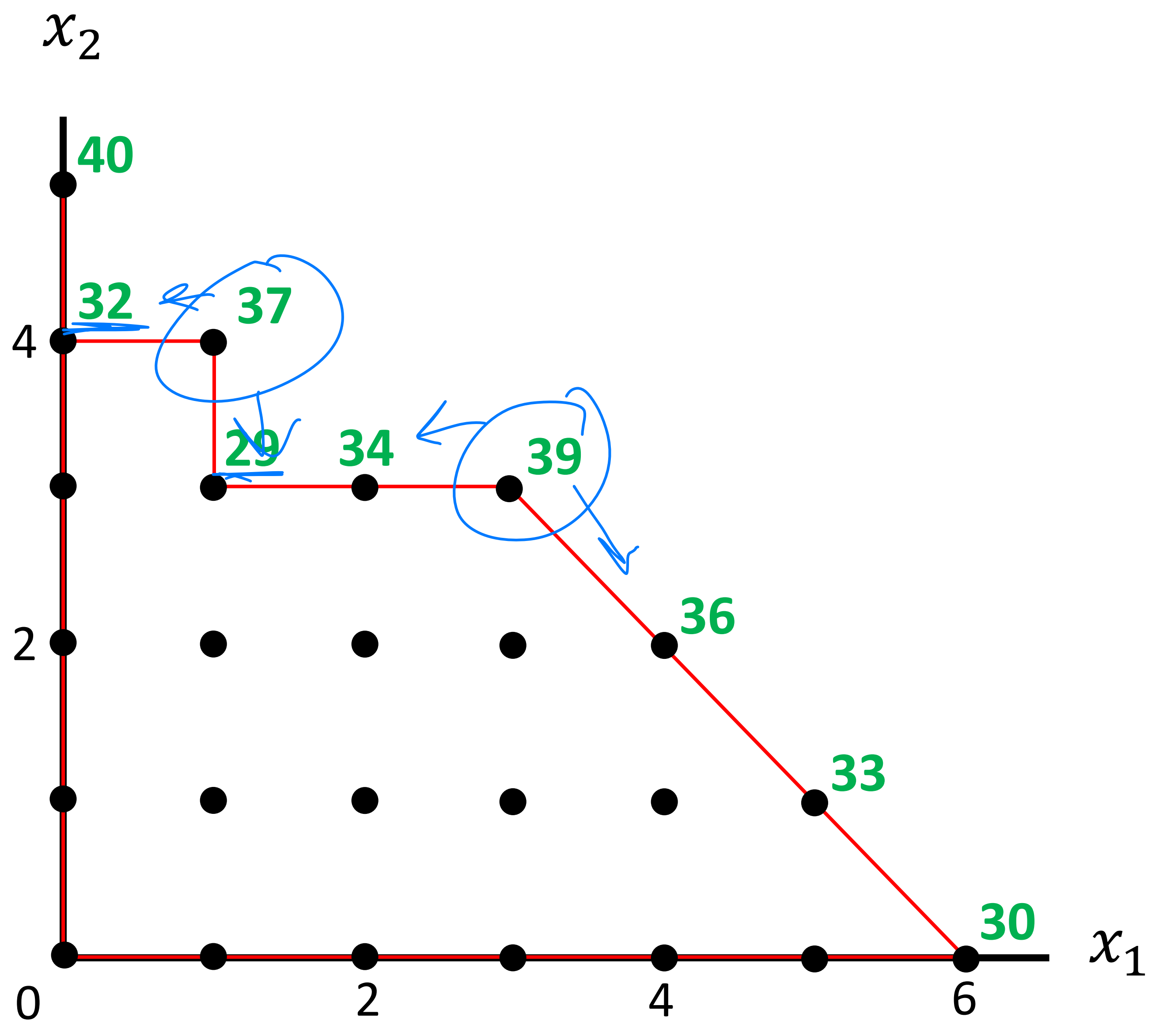
Integer feasible region:

- Not convex.



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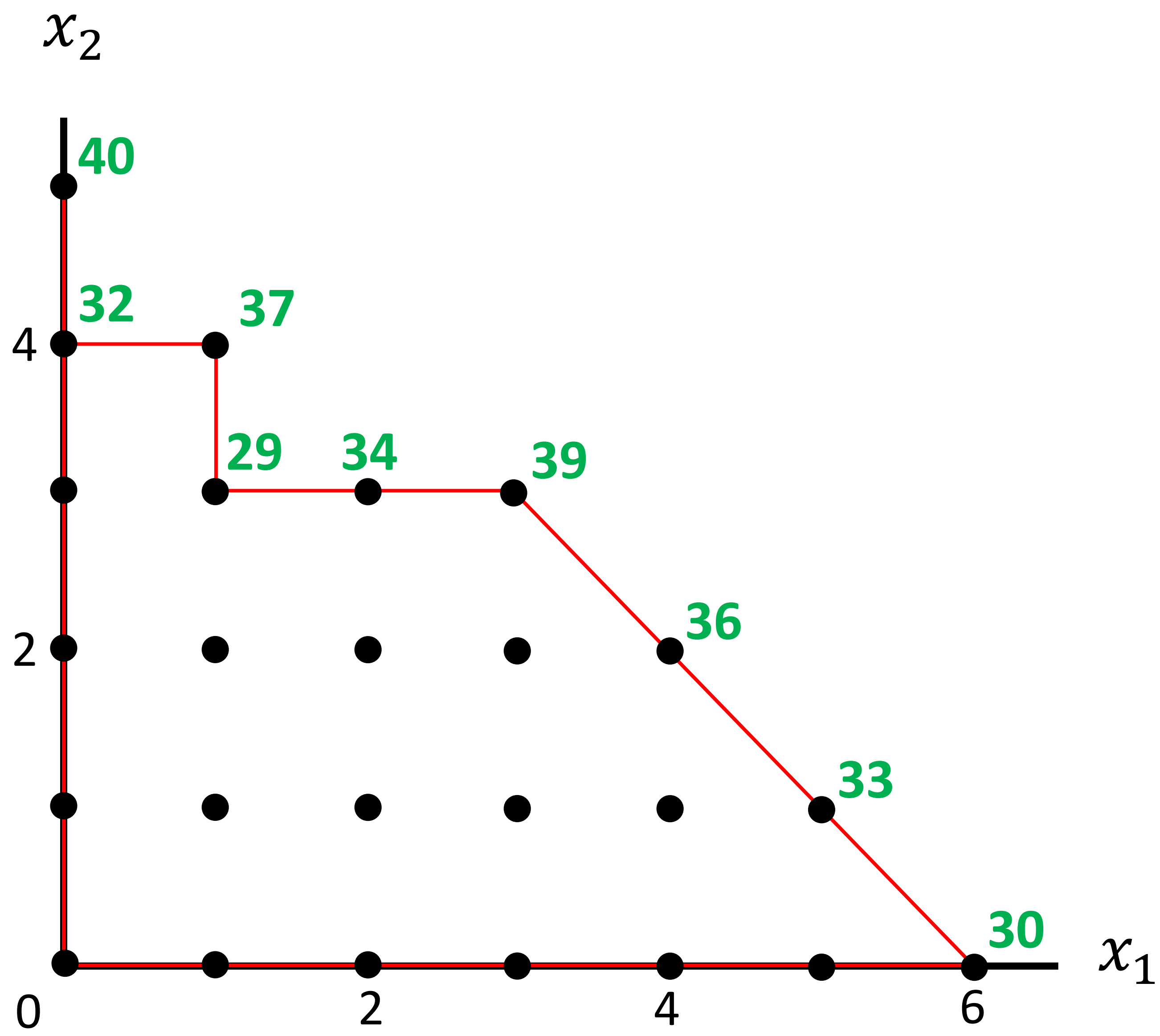

Integer feasible region:

- Not convex.
- local optimum $\neq$ global optimum.