Linear Program (LP)

x_1 = # of Rippers sold
x_2 = # of Ripper Carbons

Objective: $\max 10x_1 + 30x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Variables:

- Real numbers = solvable in polynomial time (called LP).
- Integers = not (if P≠NP) solvable in polynomial time (called integer linear program – ILP).

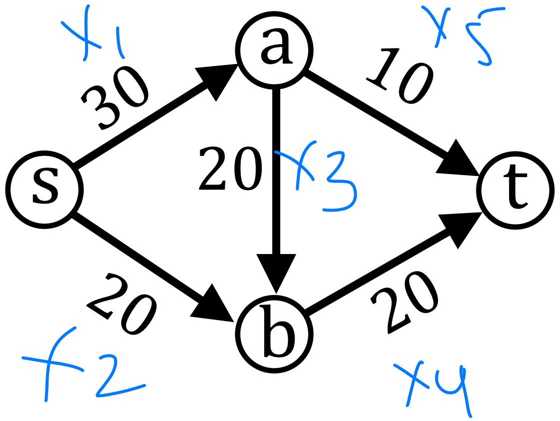
Objective:

- Can be minimization or maximization.
- Must be linear combinations of variables x_i (e.g. $a_1x_1 + \cdots + a_nx_n$ for constants a_i, not $a_ix_1x_2$).

Constraints:

- Can be $\leq$, $\geq$, $=$.
- Must be linear combinations of variables.

Maximum Flow Problem: Suppose we have the flow network below where each edge is labeled with its capacity. Give an LP whose solution is an s-t flow of maximum size.



vars: one for each edge - flow

objective: max $x_4 + x_5$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

constraints:

flow conservation constraints:

$x_3 + x_5 = x_1$

$x_4 = x_3 + x_2$

capacity constraints:

$x_1 \leq 30$

$x_2 \leq 20$

$x_3 \leq 20$

$x_4 \leq 20$

$x_5 \leq 10$

Maximum Flow Problem: Given a graph G=(V,E) with special nodes s,t and capacity function $c: E \rightarrow \mathbb{R}^{\geq 0}$, an LP for finding the maximum flow:

Maximum Flow Problem: Given a graph G=(V,E) with special nodes s,t and capacity function $c: E \to \mathbb{R}^{\geq 0}$, an LP for finding the maximum flow:

variables:

objective:

constraints:

canonical

LP Standard Form

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$

$x_1 + x_2 + x_3 \leq 40$

$2x_1 + x_3 \leq 60$

$x_1, x_2, x_3 \geq 0$

vector c

Objective: $\max c^T x$

Subject to: $Ax \leq b$

$x \geq 0$

vector x

matrix A

LP Standard Form

Objective:	$\max 10x_1 + 30x_2 + 15x_3$
Subject to:	$x_2 \le 20$
	$x_1 + x_2 + x_3 \le 40$
	$2x_1 + x_3 \le 60$
	$x_1, x_2, x_3 \ge 0$

Objective: $\max c^T x$

Subject to: $A x \le b$

$x \ge 0$

$c \cdot x$

$\begin{bmatrix} 10 \\ 30 \\ 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\max 10x_1 + 30x_2 + 15x_3$

s.t. $x_2 \le 20$

$x_1 + x_2 + x_3 \le 40$

$2x_1 + x_3 \le 60$

c^T

Objective: $\max \begin{bmatrix} 10 & 30 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

LP Standard Form

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:
$x_2 \leq 20$
$x_1 + x_2 + x_3 \leq 40$
~~$2x_1 + x_3 \geq 60$~~
$x_1, x_2, x_3 \geq 0$

Objective: max $c^T x$
Subject to: $A x \leq b$
$x \geq 0$

Objective: $\max \begin{bmatrix} 10 & 30 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Is A always a square matrix?
Silently hold up 1 for yes and 2 for no.
Then check in with neighbor.

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix}100\\300\end{pmatrix}$ $\begin{pmatrix}x_1\\x_2\end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \leq \begin{pmatrix}30\\20\\40\end{pmatrix}$$

Every LP can be turned into standard form.

1.

2.

3.

4.

LP Standard Form

Objective: $\max 100x_1 + 300x_2$

Subject to:
$$x_1 \le 30$$
$$x_2 \le 20$$
$$x_1 + x_2 \le 40$$
$$x_1, x_2 \ge 0$$

Objective: max $c^T x$, where $c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Subject to: $A\,x \le b$

$x \ge 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: ?
2. ≥ Constraints → ≤:
3. Equality Constraints → ≤:
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization:
   $$\min \alpha x_1 + \beta x_2$$
2. ≥ Constraints → ≤:
3. Equality Constraints → ≤:
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\Rightarrow$

Objective: max $c^T x$
Subject to: $A x \leq b$
$x \geq 0$

$c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.

$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$

2. $\geq$ Constraints → $\leq$:

$x \geq 5$

$-x \leq -5$

3. Equality Constraints → $\leq$:

4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

→

$\begin{pmatrix}100\\300\end{pmatrix}$ $\begin{pmatrix}x_1\\x_2\end{pmatrix}$

Objective: max $c^T x$
Subject to: $A x \le b$
$x \ge 0$

$$\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \le \begin{pmatrix}30\\20\\40\end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\ge$ Constraints → $\le$: ?
3. Equality Constraints → $\le$:
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x $\leq$ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\geq$ Constraints → $\leq$:
   $$x_1 + x_2 \geq \alpha$$
3. Equality Constraints → $\leq$:
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Objective: max $c^T x$
Subject to: $A\,x \leq b$
$x \geq 0$

$c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\geq$ Constraints → $\leq$: Negate inequality.
$$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → $\leq$:

4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$

Subject to:
$$x_1 \leq 30$$
$$x_2 \leq 20$$
$$x_1 + x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

→

Objective: max c^T x

Subject to: A x ≤ b

x ≥ 0

$$c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: Negate inequality.
   $$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → ≤:
   $$x_1 + x_2 = \alpha$$
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

Objective: $\max c^T x$, where $c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
Subject to: $A x \le b$
$x \ge 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\ge$ Constraints → $\le$: Negate inequality.
$$x_1 + x_2 \ge \alpha \rightarrow -x_1 - x_2 \le -\alpha$$
3. Equality Constraints → $\le$: Introduce $\ge$ and $\le$ constraints.
$$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \ge \alpha \text{ and } x_1 + x_2 \le \alpha$$
4. Unrestricted sign

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Objective: $\max c^T x$
Subject to: $A x \leq b$
$x \geq 0$

$$c = \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
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   $$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → $\leq$: Introduce $\geq$ and $\leq$ constraints.
   $$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \geq \alpha \text{ and } x_1 + x_2 \leq \alpha$$
4. Unrestricted sign x_1

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: Negate inequality.
   $$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → ≤: Introduce ≥ and ≤ constraints.
   $$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \geq \alpha \text{ and } x_1 + x_2 \leq \alpha$$
4. Unrestricted sign x_1: Introduce $x_1' \geq 0$, $x_1'' \geq 0$

LP Standard Form

min $5x_1 + x_2$
s.t. $x_1 = 5x_2$

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$
2. ≥ Constraints → ≤: Negate inequality.
   $x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$
3. Equality Constraints → ≤: Introduce ≥ and ≤ constraints.
   $x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \geq \alpha$ and $x_1 + x_2 \leq \alpha$
4. Unrestricted sign x_1: Introduce $x_1' \geq 0, x_1'' \geq 0$
   Change constraints $x_1 + x_2 \leq \alpha \rightarrow x_1' - x_1'' + x_2 \leq \alpha$

0 50
50 100
$x_1 = x_1' - x_1''$
-50

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \le 30$
$x_2 \le 20$
$x_1 + x_2 \le 40$
$x_1, x_2 \ge 0$

→

$\begin{pmatrix}100\\300\end{pmatrix}$ $\begin{pmatrix}x_1\\x_2\end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \le \begin{pmatrix}30\\20\\40\end{pmatrix}$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. ≥ Constraints → ≤: Negate inequality.
   $$x_1 + x_2 \ge \alpha \rightarrow -x_1 - x_2 \le -\alpha$$
3. Equality Constraints → ≤: Introduce ≥ and ≤ constraints.
   $$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \ge \alpha \text{ and } x_1 + x_2 \le \alpha$$
4. Unrestricted sign x_1: Introduce $x_1' \ge 0$, $x_1'' \ge 0$
   Change constraints $x_1 + x_2 \le \alpha \rightarrow x_1' - x_1'' + x_2 \le \alpha$

how to recover x_1?

LP Standard Form

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

$\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Objective: max c^T x
Subject to: A x ≤ b
x ≥ 0

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Every LP can be turned into standard form.

1. Minimization → Maximization: Multiply objective coefficients by -1.
   $$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$
2. $\geq$ Constraints → $\leq$: Negate inequality.
   $$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$
3. Equality Constraints → $\leq$: Introduce $\geq$ and $\leq$ constraints.
   $$x_1 + x_2 = \alpha \rightarrow x_1 + x_2 \geq \alpha \text{ and } x_1 + x_2 \leq \alpha$$
4. Unrestricted sign x_1: Introduce $x_1' \geq 0, x_1'' \geq 0$
   Change constraints $x_1 + x_2 \leq \alpha \rightarrow x_1' - x_1'' + x_2 \leq \alpha$

d = # vars
n = # constraints

how much bigger did we make our LP?

Primal and dual linear programs

Primal and dual linear programs

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Primal and dual linear programs

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Objective: $\max 100x_1 + 300x_2$

Subject to:

$x_1 \leq 30$

$x_2 \leq 20$

$x_1 + x_2 \leq 40$

$x_1, x_2 \geq 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

Objective: $\min 30y_1 + 20y_2 + 40y_3$

Subject to:

$y_1 + y_3 \geq 100$

$y_2 + y_3 \geq 300$

$y_1, y_2, y_3 \geq 0$

Primal and dual linear programs

Primal

Objective: max $c^T x$
Subject to: $A x \leq b$
$x \geq 0$

Dual

Objective: min $b^T y$
Subject to: $A^T y \geq c$
$y \geq 0$

Objective: $\max 100x_1 + 300x_2$
Subject to: $x_1 \leq 30$
$x_2 \leq 20$
$x_1 + x_2 \leq 40$
$x_1, x_2 \geq 0$

Objective: $\min 30y_1 + 20y_2 + 40y_3$
Subject to: $y_1 + y_3 \geq 100$
$y_2 + y_3 \geq 300$
$y_1, y_2, y_3 \geq 0$

Primal and dual linear programs

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Strong Duality Theorem: If x^* is an optimal solution for the primal LP, then there is an optimal solution y^* for its dual such that

$$c \cdot x^* = y^* \cdot b.$$

x_1 = # of Rippers sold in a day
x_2 = # of Ripper Carbons sold in a day
x_3 = # of Ripper Jrs sold in a day

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:
$x_2 \leq 20$
$x_1 + x_2 + x_3 \leq 40$
$2x_1 + x_3 \leq 60$
$x_1, x_2, x_3 \geq 0$

x_1 = # of Rippers sold in a day
x_2 = # of Ripper Carbons sold in a day
x_3 = # of Ripper Jrs sold in a day

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$$x_2 \leq 20$$
$$x_1 + x_2 + x_3 \leq 40$$
$$2x_1 + x_3 \leq 60$$
$$x_1, x_2, x_3 \geq 0$$

Linear combinations of constraints are also valid constraints!

x_1 = # of Rippers sold in a day
x_2 = # of Ripper Carbons sold in a day
x_3 = # of Ripper Jrs sold in a day

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

$x_1, x_2, x_3 \geq 0$

Linear combinations of constraints are also valid constraints!

2A : $2x_2 \leq 40$

A+B

$x_2 \leq 20$

$x_1 + x_2 + x_3 \leq 40$

$x_2 + 2x_2 + x_3 \leq 60$

15A + 15B : $15x_2 \leq 300$

$15x_1 + 15x_2 + 15x_3 \leq 600$

$15x_1 + 30x_2 + 15x_3 \leq 900$

$10x_1 + 30x_2 + 15x_3 \leq 900$

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$(y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$10x_1 + 30x_2 + 15x_3 \leq 20y_1 + 40y_2 + 60y_3,$$

If:

$$y_2 + 2y_3 \geq 10$$
$$y_1 + y_2 \geq 30$$
$$y_2 + y_3 \geq 15$$
$$y_1, y_2, y_3 \geq 0$$

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$(y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \leq 20y_1 + 40y_2 + 60y_3$$

$$10x_1 + 30x_2 + 15x_3 \leq 20y_1 + 40y_2 + 60y_3,$$

If:

$$y_2 + 2y_3 \geq 10$$
$$y_1 + y_2 \geq 30$$
$$y_2 + y_3 \geq 15$$
$$y_1, y_2, y_3 \geq 0$$

Need to find valid y_i's.

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

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If:

$$y_2 + 2y_3 \geq 10$$
$$y_1 + y_2 \geq 30$$
$$y_2 + y_3 \geq 15$$
$$y_1, y_2, y_3 \geq 0$$

Need to find valid y_i's.

$y_1 = 10, y_2 = 20, y_3 = 10 \Longrightarrow$ objective ≤ 1600

Objective:	$\max 10x_1 + 30x_2 + 15x_3$	
Subject to:	$x_2 \leq 20$	A
	$x_1 + x_2 + x_3 \leq 40$	B
	$2x_1 + x_3 \leq 60$	C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
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y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \leq 20y_1 + 40y_2 + 60y_3$$

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If:

$$y_2 + 2y_3 \geq 10$$
$$y_1 + y_2 \geq 30$$
$$y_2 + y_3 \geq 15$$
$$y_1, y_2, y_3 \geq 0$$

~~Need to find valid y_i's.~~

Need to find best y_i's.

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \leq 20$ A

$x_1 + x_2 + x_3 \leq 40$ B

$2x_1 + x_3 \leq 60$ C

Multiplier	Constraint
y_1	$x_2 \leq 20$
y_2	$x_1 + x_2 + x_3 \leq 40$
y_3	$2x_1 + x_3 \leq 60$

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2$

$(y_2 + 2y_3)x_1 + ($

Linear Programmize™ It!

$y_2 + 2y_3 \geq 10$

$y_1 + y_2 \geq 30$

$y_2 + y_3 \geq 15$

$y_1, y_2, y_3 \geq 0$

~~Ne ... y_i's.~~

Need to find best y_i's.

Objective: $\max 10x_1 + 30x_2 + 15x_3$

Subject to:

$x_2 \le 20$ A

$x_1 + x_2 + x_3 \le 40$ B

$2x_1 + x_3 \le 60$ C

Objective:

Subject to:

y_1(Constraint_A) + y_2(Constraint_B) + y_3(Constraint_C)

$$y_1x_2 + y_2x_1 + y_2x_2 + y_2x_3 + 2y_3x_1 + y_3x_3 \le 20y_1 + 40y_2 + 60y_3$$

$$(y_2 + 2y_3)x_1 + (y_1 + y_2)x_2 + (y_2 + y_3)x_3 \le 20y_1 + 40y_2 + 60y_3$$

$$10x_1 + 30x_2 + 15x_3 \le 20y_1 + 40y_2 + 60y_3,$$

If:

$$y_2 + 2y_3 \ge 10$$
$$y_1 + y_2 \ge 30$$
$$y_2 + y_3 \ge 15$$
$$y_1, y_2, y_3 \ge 0$$

~~Need to find valid y_i's.~~

Need to find best y_i's.

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Objective: $\max \begin{bmatrix} 10 & 30 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Objective: $\min \begin{bmatrix} 20 & 40 & 60 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Subject to: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 30 \\ 15 \end{bmatrix}$

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Dual

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof: ?

Dual

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof:

Objective: $\min b^T y$

Subject to: $A^T y \geq c$ →

$y \geq 0$

Standard Form

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: The dual of a dual is the original primal.

Proof:

Objective: min $b^T y$
Subject to: $A^T y \geq c$
$y \geq 0$

→ (Standard Form)

Objective: max $-b^T y$
Subject to: $-A^T y \leq -c$
$y \geq 0$

→ (Dual)

Objective: min $-c^T z$
Subject to: $-{A^T}^T z \geq -b$
$z \geq 0$

→ (Standard Form)

Objective: max $c^T z$
Subject to: $A z \leq b$
$z \geq 0$

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$c^T \bar{x} \leq$

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$$c^T \bar{x} \leq (A^T \bar{y})^T \bar{x}$$

Since $A^T y \geq c$

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$$c^T \bar{x} \leq (A^T \bar{y})^T \bar{x} = (\bar{y}^T A)\, \bar{x}$$

Since transpose of multiplication is multiplication of transposes (reversed)

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\overline{x}$ is any feasible solution to the primal and $\overline{y}$ is any feasible solution to the dual, then $c^T \overline{x} \leq b^T \overline{y}$.

Proof:

$$c^T \overline{x} \leq (A^T \overline{y})^T \overline{x} = (\overline{y}^T A)\, \overline{x} = \overline{y}^T (A\, \overline{x})$$

Matrix multiplication is associative.

Dual

Primal

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: $\min b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$$c^T \bar{x} \leq (A^T \bar{y})^T \bar{x} = (\bar{y}^T A)\, \bar{x} = \bar{y}^T (A\, \bar{x}) \leq \bar{y}^T b$$

Since $A x \leq b$

Dual

Primal

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Dual

Objective: min $b^T y$

Subject to: $A^T y \geq c$

$y \geq 0$

Theorem: If $\bar{x}$ is any feasible solution to the primal and $\bar{y}$ is any feasible solution to the dual, then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$$c^T \bar{x} \leq (A^T \bar{y})^T \bar{x} = (\bar{y}^T A)\, \bar{x} = \bar{y}^T (A\, \bar{x}) \leq \bar{y}^T b = b^T \bar{y}$$

Since b and $\bar{y}$ are 1-dimensional vectors.