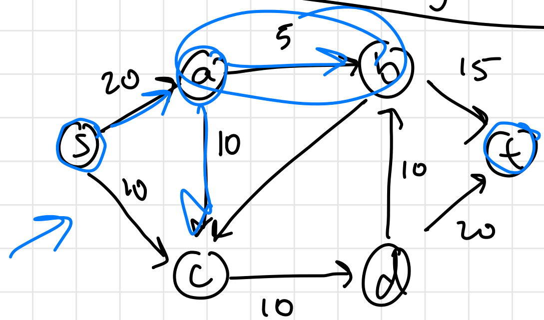
Max Flow / Min cut

Input: directed graph $G = (V, E)$
$s, t \in V$

capacity function $c: E \to \mathbb{R}^{\geq 0}$



Max flow: (Soviet problem - how many trains can I send b/w EG and Moscow)

A function $f: E \to \mathbb{R}^{\geq 0}$ such that

① For all $v \in V \setminus \{s,t\}$, $\sum_u f(u \to v) = \sum_w f(v \to w)$

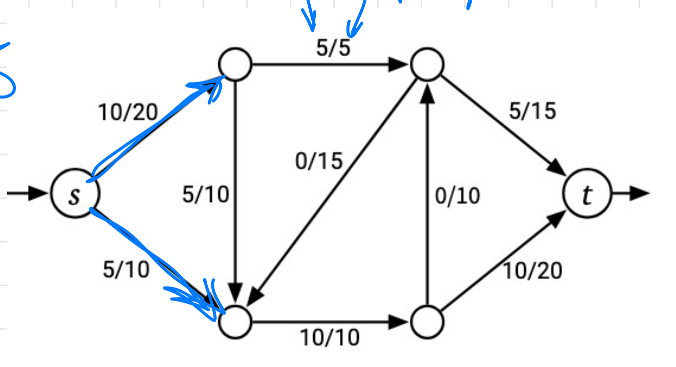
conservation of flow

② For all $e \in E$, $0 \leq f(e) \leq c(e)$

feasibility

③ $|f| = \sum_v f(s \to v)$ is maximized.

$|f| = 10+5$
$= 15$



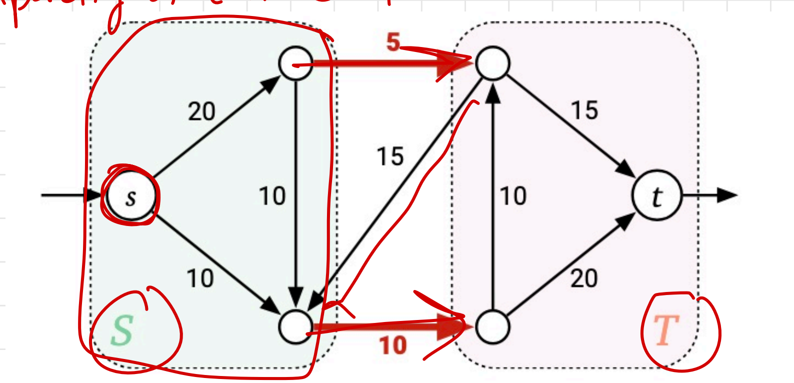
Min cut: (US prob — disconnect EG from moscow)

A partition S,T of V such that

① $s \in S$, $t \in T$

② $\|S,T\| = \sum_{v\in S} \sum_{w \in T} c(v \to w)$ is minimized.

"capacity of cut S-T"



$\|S,T\| = 5+10$
$= 15$

Observation: $|f| = \|S,T\|$ for optimal f, S,T so far

"Easy" (book): $\underset{\max}{|f|} \le \underset{\min}{\|S,T\|}$ for all feasible flows f and all s-t cuts S,T

If $|f| = \|S,T\|$ is best we can do

Not "Easy": do such a f and S,T always exist? How to find them?

The Max Flow-Min Cut Theorem: In every flow network, the value of the max s-t flow is equal to the value of a min s-t cut.

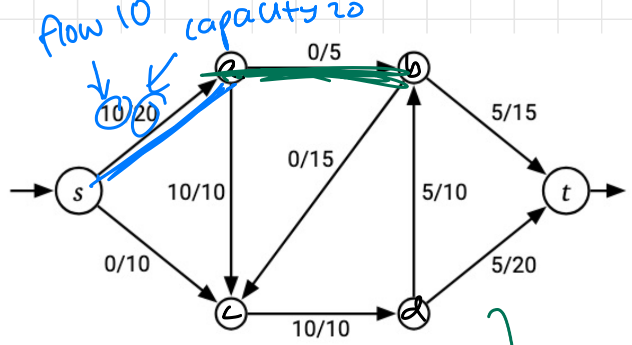
Proof (detail in notes): Given a feasible flow, it can either

1) be transformed into a greater feasible flow, or
2) is equal to the value of a min cut.

Residual Graph + residual capacity

Let f be any feasible flow. Define

$$\underset{\text{"residual capacity of } f\text{"}}{c_f} = \begin{cases} \underline{c(u \to v) - f(u \to v)} & \text{if } u \to v \in E \\ \underline{f(u \to v)} & \text{if } v \to u \in E \end{cases}$$



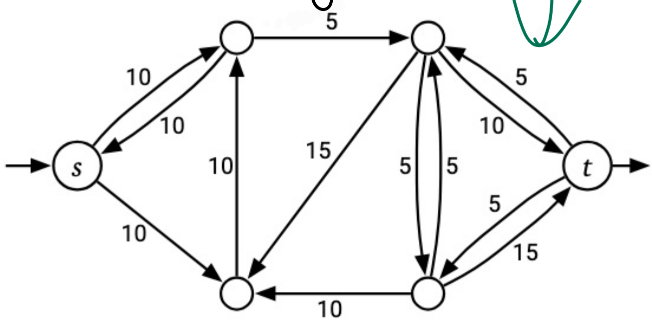
$s \to a \in E$

$c_f(s \to a) = c(s \to a) - f(s \to a)$

$a \to s \notin E$, but reverse

$c_f(a \to s) = f(a \to s)$

residual graph

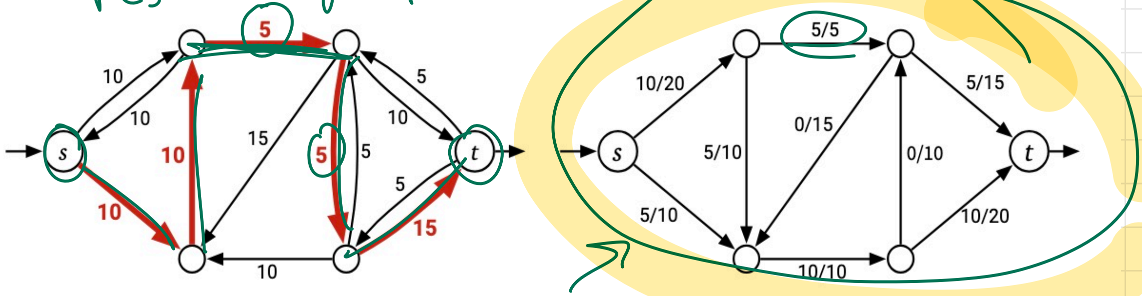


$c_f(a \to c)$
$c_f(c \to a)$

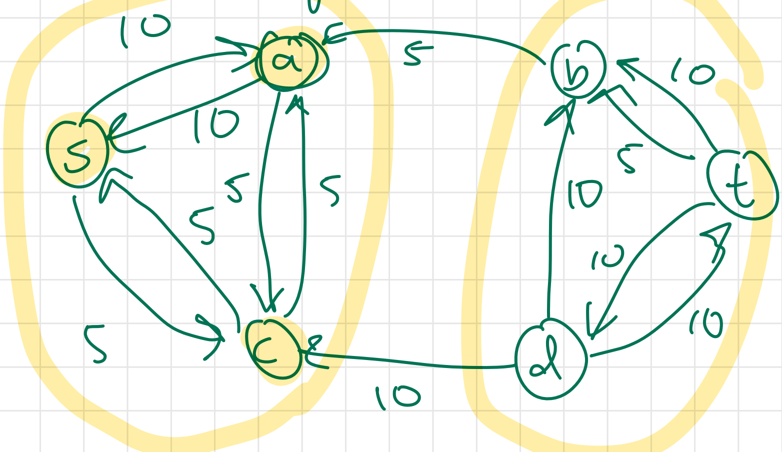
$c_f(s \to c)$
$c_f(c \to s)$

$c_f(b \to t)$
$c_f(t \to b)$

Augmenting path: s-t path in residual graph residual graph



draw new residual graph and look for an augmenting path.



optimal f

f^*

S

S^* = nodes reachable from s in residual graph

$T^* = V \setminus S$

Ford-Fulkerson(G, c):

f ← 0 on all edges

G_r ← residual graph based on f

O(E)

while there is an augmenting path

P in G_r:

$F \leftarrow \min c_f$ along P

$f \leftarrow f + F$ on all edges along P

update G_r

$O(E)$ — $|f^*|$ times

return f

$O(E|f^*|)$

Ford-Fulkerson	:	1954	$O(E\|f^*\|)$
Orlin	:	2012	$O(EV)$
Chen et al.	:	2022	$O(E^{1+o(1)})$

strictly more than constant