

Given a language L , is there a TM that decides it?

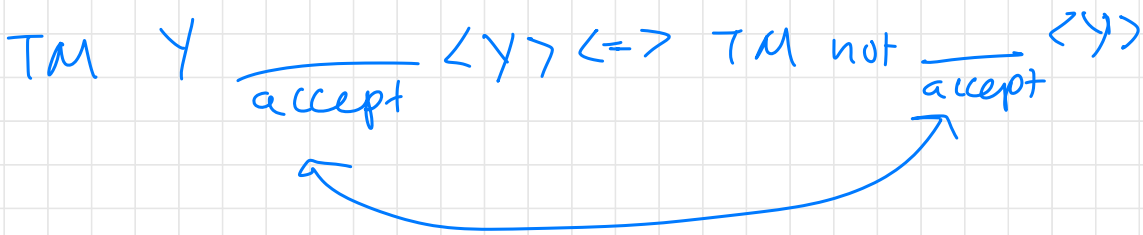
If not, L is undecidable.

Examples of undecidable languages:

- SELF REJECT = $\{ \langle M \rangle : M \text{ rejects } \langle M \rangle \}$
- SELF HALT = $\{ \langle M \rangle : M \text{ halts on } \langle M \rangle \}$
- SELF ACCEPT = $\{ \langle M \rangle : M \text{ accept } \langle M \rangle \}$
- ACCEPT = $\{ \langle M, w \rangle : M \text{ accepts } w \}$

How do we prove a language is undecidable?

(1) Diagonalization — cantor proved that $|\mathbb{R}| > |\mathbb{Z}|$



(2) reduction from deciding a known undecidable language

Theorem HALT = $\{ \langle M, w \rangle : M \text{ halts on } w \}$
is undecidable.

Proof We prove that HALT is undecidable by reduction from deciding SELFHALT.

Suppose HALT is decidable. Let H be the TM that decides HALT.

Now we can make a TM called SH that decides SELFHALT as follows:

SH($\langle M \rangle$):
run H on $\langle M, \langle M \rangle \rangle$

data type check:

SH is a TM that accepts $\langle M \rangle$ iff
 M halts on $\langle M \rangle$
It is a TM that accepts $\langle M, w \rangle$ iff
 M halts on w
 M is a TM that is arbitrary

SH accepts $\langle M \rangle$ iff M halts on $\langle M \rangle$.

- SH accepts $\langle M \rangle$ if M halts on $\langle M \rangle$:

M halts on $\langle M \rangle$.

H accepts on $\langle M, \langle M \rangle \rangle$.

SH accepts on $\langle M \rangle$

- M halts on $\langle M \rangle$ if SH accepts $\langle M \rangle$;

SH rejects $\langle M \rangle$ if M does not halt on $\langle M \rangle$;

contrapositive

$$p \Rightarrow q$$

$$\neg q \Rightarrow \neg p$$

M does not halt on $\langle M \rangle$.

So H rejects $\langle M, \langle M \rangle \rangle$

So SH rejects $\langle M \rangle$.

But SH is undecidable!

So H cannot exist.

- Most languages of form

"Given $\langle M \rangle$, does M accept ___?"

are undecidable

- are there undecidable problems that are not about TMs that are undecidable?

1) Diophantine Equations - 1970

Given a polynomial, does it have integer solution?

$$a^2 + b^2 = c^2$$

yes infinitely many

$$a^n + b^n = c^n \quad n > 2$$

2) Post Correspondence Problem