Prove that each of the following languages is ***not*** regular, first using fooling sets and then (for problems 3, 4, and 5) using a reduction argument. You may use the fact (proven in class and in the lecture notes) that the language $\{0^n 1^n \mid n \geq 0\}$ is not regular. See the next page for a solved example showing both types of proof.

1. $\left\{0^{2^n} \,\middle|\, n \geq 0\right\}$

2. $\{0^{2n} 1^n \mid n \geq 0\}$

3. $\{0^m 1^n \mid m \neq 2n\}$

   [Hint: There is a short reduction argument, but write the fooling set argument first.]

4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

   [Hint: There is a short reduction argument, but write the fooling set argument first.]

5. Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ([)] is not, because the left and right delimiters don't match.

   [Hint: There is a short reduction argument, but write the fooling set argument first.]

Harder problems to think about later:

6. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0, 1\}^*$, and some pair of substrings w_i and w_j are equal.

7. $\left\{0^{n^2} \,\middle|\, n \geq 0\right\}$

*8. $\{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\}$

Solved problem:

9. Prove that the language $L = \{w \in (0+1)^* \mid \#(0,w) = \#(1,w)\}$ is ***not*** regular.

Solution (fooling set 0^*):

Consider the infinite set $F = \{0^n \mid n \geq 0\}$, or more simply $F = 0^*$.

We claim that every pair of distinct strings in F has a distinguishing suffix.

> Let x and y be arbitrary distinct strings in F.
> The definition of F implies $x = 0^i$ and $y = 0^j$ for some integers $i \neq j$.
> Let z be the string 1^i.
> Then $xz = 0^i1^i \in L$.
> But $yz = 0^j1^i \notin L$, because $i \neq j$.
> So z is a distinguishing suffix for x and y.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

*This is **exactly** the proof from the lecture notes for the canonical non-regular language $\{0^n1^n \mid n \geq 0\}$. The inner box is a proof that every pair of district strings in F has a distinguishing suffix.*

Solution (fooling set 0^*):

For any natural number n, let $x_n = 0^n$, and let $F = \{x_n \mid n \geq 0\} = 0^*$.

Let i and j be arbitrary distinct natural numbers.

Let z_{ij} be the string 1^i.

Then $x_i z_{ij} = 0^i1^i \in L$.

But $x_j z_{ij} = 0^j1^i \notin L$, because $i \neq j$.

So z_{ij} is a distinguishing suffix for x_i and x_j.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

*This is another way of writing exactly the same proof that emphasizes the counter intuition; any algorithm that recognizes L **must** count 0s.*

Solution (reduction via closure): For the sake of argument, suppose L is regular.

Then the language $L \cap 0^*1^* = \{0^n1^n \mid n \geq 0\}$ would also be regular, because regular languages are closed under intersection.

But we proved in class that $\{0^n1^n \mid n \geq 0\}$ is not regular; we've reached a contradiction.

We conclude that L cannot be regular.

*And this is **why** the proof for $\{0^n1^n \mid n \geq 0\}$ also works verbatim for this language.* ■